NAVIGATION

AND

NAUTICAL ASTRONOMY.

COFFIN.

John Huntington Crane Coffin

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NAUTICAL ASTRONOMY.

PREPARED FOR THE USE OF THE U. S. NAVAL ACADEMY.

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NOTICE.

This Treatise was originally prepared by Prof. Chauvenet to be used in manuscript by the students of the Naval Academy. With Bowditch's Navigator, oral instruction, the use of instruments, and computation of examples, it constituted the course of instruction in Navigation and Nautical Astronomy.

In its printed form, some subjects are more fully discussed, others introduced, and various suggestions given on points of practice.

In this edition examples are supplied, which will serve both as illustrations, and as forms for the arrangement of computations. Those in Nautical Astronomy are mainly adapted to the Ephemeris for 1865.

It has been my purpose, as I should find time from incessant official duties, to prepare a more complete work, or to supplement it with a treatise on the practice of Navigation.

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Prof. of Astronomy, Navigation, and Surveying.

Naval Academy, August 1, 1865.

GREEK LETTERS.

A	α	Alpha,	N	ν	Nu,
B	β	Beta,	Ξ	ξ	Xi,
Γ	γ	Gamma,	O	o	Omicron,
Δ	δ	Delta,	Π	π	Pi,
E	ε	Epsilon,	P	ρ	Rho,
Z	ζ	Zeta,	Σ	σ	Sigma,
H	η	Eta,	T	τ	Tau,
Θ	θ	Theta,	Υ	υ	Upsilon,
I	ι	Iota,	Φ	φ	Phi,
K	$\varkappa$	Kappa,	X	χ	Chi,
Λ	λ	Lambda,	Ψ	ψ	Psi,
M	μ	Mu,	Ω	ω	Omega.

CONTENTS.

NAVIGATION.

CHAPTER I.

THE SAILINGS.

PLANE SAILING.

1. SUPPOSE the compass-needle constantly to point to the north, a ship which is steered by it upon any given course must *cross every meridian at the same angle*, namely, the angle given by the compass. She does not sail on a great circle, except when she sails on the equator, east or west, or on a meridian, north or south. All other great circles intersect successive meridians at varying angles.

A line which makes the same angle with each successive meridian is called a *loxodromic curve;* in old nautical works, a *rhumb-line;* more commonly, the *ship's track.*

The constant angle which it makes with the meridian is the *course*, and is called the *true course*, to distinguish it from the *compass course.*

The length of the line considered, or the distance sailed, is called the *distance.*

The corresponding increase or decrease of latitude is the *difference of latitude.*

The distance between the meridian left, and that arrived at, measured on a parallel of latitude, is the *departure* on that parallel.

The distance between these meridians, measured on the equator, is the corresponding *difference of longitude.*

2. The following notation will be employed; the references being to Fig. 1, in which C A represents a portion of a loxodromic curve:

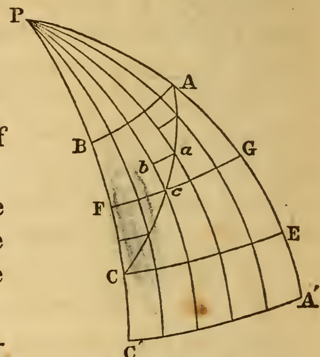

Fig. 1.

$C =$ B C A, the course.
$d =$ C A, the distance.
$l =$ C B $=$ E A, the difference of latitude.
$p =$ the departure: C E in the latitude of C, B A in the latitude of B, F G in the latitude of F.
$D =$ C′ A′, the difference of longitude.

$L =$ C′ C, the latitude left, } + when North.
$L' =$ A′ A, the latitude arrived at, } − when South.
$\lambda =$ the longitude left, } + when West.
$\lambda' =$ the longitude arrived at, } − when East.

$$\left.\begin{array}{lll}\text{Evidently} & l = L' - L, & D = \lambda' - \lambda; \\ \text{whence} & L' = L + l, & \lambda' = \lambda + D,\end{array}\right\} \qquad (1)$$

in which attention must be paid to the signs, or names.* These formulas accord with the precepts on page 50 of Bowditch's Navigator.

3. If the distance is very small, so that the curvature of the earth may be neglected, then C A may be regarded as a right line, and the triangle C A B as a right plane triangle. From this we have

$$\cos C = \frac{l}{d}, \qquad \sin C = \frac{p}{d}, \qquad \tan C = \frac{p}{l}; \qquad (2)$$

or,

$$l = d \cos C, \qquad p = d \sin C, \qquad p = l \tan C, \qquad (3)$$

* If N. and W. are regarded as positive, S. and E. are negative, and may be treated as such, without the formality of substituting the signs + and −.

in which p is the departure in the latitude of C or A; indifferently, as their distance is very small.

The *Traverse Table*, or *Table of Right Triangles*, contains l and p for different values of C and d. Table I. in Bowditch's Navigator contains l and p for each unit of d from 1 to 300, and for each quarter-point of C. Table II. contains them for each unit of d and each degree of C.

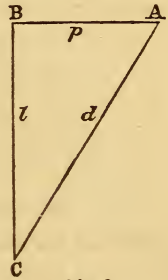

Fig. 2.

These quantities form a plane right triangle (Fig. 2), in which

d is the hypothenuse,
C one of the angles,
l the side adjacent } that angle.
p " opposite }

In the Tables, the columns of *distance*, *difference of latitude*, and *departure*, might be appropriately headed, respectively, *hypothenuse*, *side adjacent*, and *side opposite*.

4. The first two of equations (3) afford the solution of the most common elementary problem of navigation and surveying, viz.:

Problem 1. *Given the course and distance, to find the difference of latitude and departure, the distance being so small that the curvature of the earth may be neglected.*

These equations also afford solutions of all the cases of Plane Sailing. (Bowd., pp. 52–58.)*

5. Problem 2. *Given the course and distance, to find the difference of latitude and departure, when the distance is so great that the curvature of the earth cannot be neglected.*

Solution. Let the distance C A (Fig. 1) be divided into parts, each so small that the curvature of the earth may be neglected in computing its corresponding difference of latitude and departure.

* The first and sixth are the most important.

For each such small distance, as $c\ a$,

$$l = d \cos C, \qquad\qquad p = d \sin C.$$

Representing the several partial distances by d_1, d_2, d_3, &c., the corresponding values of l and p by l_1, l_2, l_3, &c., and p_1, p_2, p_3, &c., and the sums respectively by $[d]$, $[l]$, $[p]$, we have

$$l_1 + l_2 + l_3 + \text{\&c.} = (d_1 + d_2 + d_3 \text{ \&c.}) \cos C,$$
$$p_1 + p_2 + p_3 + \text{\&c.} = (d_1 + d_2 + d_3 \text{ \&c.}) \sin C;$$

or,

$$[l] = [d] \cos C,$$
$$[p] = [d] \sin C.$$

Since the distance between two parallels of latitude is the same on all meridians, the sum of the several partial differences of latitude will be the whole difference of latitude; As in Fig. 1.

C B = E A = the sum of all the sides, $c\ b$, of the small triangles;

and we shall have generally, as in Prob. 1, whatever the distance, d,

$$l = d \cos C.$$

We also have

$$p = d \sin C,$$

if we regard p as the *sum of the partial departures*, each being taken in the latitude of its triangle; so that the difference of latitude and departure are calculated by the same formulas, when the curvature of the earth is taken into account, as when the distance is so small that the curvature may be disregarded; or, in other words, *as if the earth were a plane.*

But the sum of these partial departures, $b\ a$ of Fig. 1, is evidently less than C E, the distance between the meridians left and arrived at on the parallel C E, which is nearest the equator; and greater than B A, the distance of these meridians on the parallel B A, which is farthest from the equa-

tor. But it is *nearly* equal to F G, the distance of these meridians on a middle parallel between C and A; and *exactly* equal to the distance on a parallel a little nearer the pole, and whose precise position will be subsequently determined. (See Problem 10, Mercator's Sailing.)

We take then $L_0 = \frac{1}{2}(L + L')$, or, more exactly,

$$L_0 = \frac{1}{2}(L + L') + \varDelta L,$$

as the latitude for the departure, p.

6. *Middle Latitude Sailing* regards the departure, p, as the distance between the meridian left and that arrived at on the middle parallel of latitude; or takes $L_0 = \frac{1}{2}(L + L')$.

TRAVERSE SAILING.

7. If the ship sail on several courses, instead of a single course, she describes an irregular track, which is called a *Traverse.*

PROBLEM 3. *To reduce several courses and distances to a single course and distance, and find the corresponding differences of latitude and departure.*

Solution. If in Fig. 1 we regard C as different for each partial triangle, and represent the several courses by C_1, C_2, C_3, &c., we evidently have

$$\begin{array}{ll} l_1 = d_1 \cos C_1, & p_1 = d_1 \sin C_1, \\ l_2 = d_2 \cos C_2, & p_2 = d_2 \sin C_2, \\ l_3 = d_3 \cos C_3, & p_3 = d_3 \sin C_3, \\ \quad \&\text{c.} & \quad \&\text{c.} \end{array}$$

and $[l] = l_1 + l_2 + l_3$, &c., $\quad [p] = p_1 + p_2 + p_3$, &c.;

or, as in the more simple case of a single course,

The whole difference of latitude is equal to the sum of the partial differences of latitude;

The whole departure is equal to the sum of the partial departures.

This applies to all cases, if we use the word *sum* in its general or algebraic sense.

If we represent by L_n the sum of the northern diffs. of latitude,
" " L_s " " southern " "
" " P_w " " western departures,
" " P_e " " eastern "

we have as the *arithmetical* formulas,

$$[l] = L_n \sim L_s \text{ of the same name as the greater,}$$
$$[p] = P_w \sim P_e \quad \text{" " " " "}$$

which accord with the usual rules. (Bowd., p. 59 and p. 264.)

The *Traverse Form* (p. 60 and pp. 266 to 286) facilitates the computation.

The course, C, and distance, $[d]$, corresponding to $[l]$ and $[p]$, may be found *nearly* by Plane Sailing.*

8. The departure may be regarded as measured on the middle parallel, either between the extreme parallels of the traverse, or between that of the latitude left and that arrived at. In a very irregular traverse it is difficult to determine the precise parallel; but, except near the pole, and for

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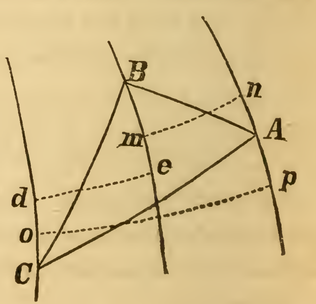
* C and $[d]$ are not accurately found, because $[p]$, the sum of the partial departures of the traverse, is not the same as p, the departure of the loxodromic curve connecting the extremities of the traverse. Thus suppose a ship to sail from C to A by the traverse C B, B A, her departure will be by traverse sailing $d\,e + m\,n$; whereas, if the ship sail directly from C to A, the departure will be $o\,p$, which is greater or less than $d\,e + m\,n$, according as it is nearer to, or farther from the equator. Thus we should obtain in the two cases a different course and distance between the same two points. In ordinary practice, however, such difference is immaterial.



a distance exceeding an ordinary day's run, the middle latitude suffices. (Bowd., p. 59, note.)

It is easy, however, to separate a traverse into two or more portions, and compute for each separately.

PARALLEL SAILING.

9. The relations of the quantities C, d, l, and p are expressed in equations (3). When the difference of longitude also enters, then some further considerations are necessary.

PROBLEM 4. *To find the relations between*

L, *the latitude of a parallel,*

p, *the departure of two meridians on that parallel,* and

D, *the corresponding difference of longitude.*

Solution. In Fig. 3, let

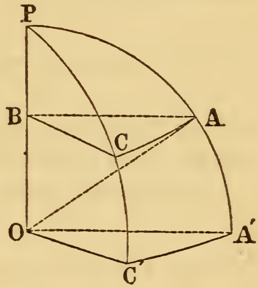


Fig. 3.

P A A′, P C C′ be two meridians.

A C $= p$, their departure on the parallel A C, whose latitude is A O A′ $=$ O A B $= L$, and whose radius is B A $= r$.

A′ C′ $= D$, the measure of A P C, the difference of longitude of the same meridians, on the equator A′ C′, whose radius is O A′$=$O A $= R$.

A C, A′ C′ are similar arcs of two circles, and are therefore to each other as the radii of those circles; that is,

$$AC : A'C' = BA : OA', \qquad \text{or} \qquad p : D = r : R.$$

In the right triangle O B A,

$$BA = OA \times \cos OAB, \qquad \text{or} \qquad r = R \cos L; \quad (4)$$

that is, *the radius of a parallel of latitude is equal to the radius of the equator multiplied by the cosine of the latitude.*

Substituting (4) in the preceding proportion, we obtain

$$p : D = \cos L : 1,$$

or

$$p = D \cos L, \qquad D = p \sec L, \qquad (5)$$

which express the relations required. (Bowd., p. 63.)

These relations may be graphically represented by a right plane triangle (Fig. 4), of which

D is the hypothenuse,

L, one of the angles,

p, the side adjacent that angle.

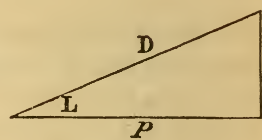


Fig. 4.

The *Traverse Table*, or *Table of Right Triangles*, may therefore be used for the computation (Bowd., p. 65, "by inspection").

MIDDLE LATITUDE SAILING.

10. Problem 5. *Given the course and distance and the latitude left, to find the difference of longitude.*

Solution. By plane sailing,

$$l = d \cos C, \qquad p = d \sin C; \qquad (3)$$

by Arts. 2 and 6,

$$L' = L + l, \qquad L_0 = \tfrac{1}{2}(L' + L) = L + \tfrac{1}{2} l; \qquad (6)$$

and by equation (5),

$$D = p \sec L_0, \qquad (7)$$

or

$$D = d \sin C \sec L_0. \qquad (8)$$

Equations (3), (6), and (7) or (8) afford the solution required.

The assumption of $L_0 = \frac{1}{2}(L' + L)$, or the *middle latitude*, suffices for the ordinary distance of a day's run; but for larger distances, and where precision is required, we must take (Art. 5)

$$L_0 = \tfrac{1}{2}(L + L') + \varDelta L, \tag{7}$$

in which $\varDelta L$ is a small correction to be added numerically to the middle latitude. A formula for computing it is given in Prob. 10, under "Mercator's Sailing." Its value in the most common cases is given in Bowd., p. 76, and in Stanley's Tables, p. 338.

11. Strictly, the middle latitude should be used only when both latitudes, L and L', are of the same name, as is evident from Fig. 1.

If these latitudes are of different names, and the distance is small, $\frac{1}{2}(L+L')$, numerically, may be used; or we may even take $p = D$, since the meridians near the equator are sensibly parallel.

If the distance is great, the two portions of the track on different sides of the equator may be treated separately. Thus, in Fig. 5, the track C A is separated by the equator into two parts, C E and E A.

For C E, we have

$$\text{C C}' = l_1 = -L,$$
$$p_1 = -L \tan C,$$
$$\text{C}'\,\text{E} = D_1 = p_1 \sec \tfrac{1}{2} L,$$
$$= -L \tan C \sec \tfrac{1}{2} L \text{ nearly.}$$

For E A, we have

$$\text{A}'\,\text{A} = l_2 = L',$$
$$p_2 = L' \tan C,$$
$$\text{E A}' = D_2 = p_2 \sec \tfrac{1}{2} L',$$
$$= L' \tan C \sec \tfrac{1}{2} L' \text{ nearly.}$$

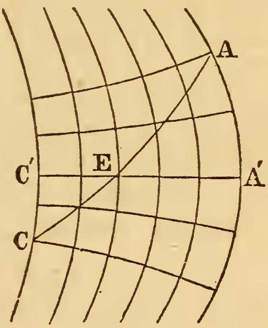


Fig. 5.

Whence we obtain C′ A′ or $D = D_1 + D_2$.

Instead of the middle latitudes $\frac{1}{2} L$ and $\frac{1}{2} L'$, we may use more rigidly $(\frac{1}{2} L + \varDelta L)$ and $(\frac{1}{2} L' + \varDelta L')$.

When several courses and distances are sailed, as is ordinarily the case in a day's run, p and l are found as in trav-

erse sailing, and then D by regarding p as on some parallel midway between the extremes of the traverse. (Art. 8.) (Bowd., p. 264.)

12. The relations of the quantities involved in middle latitude sailing, namely,

$$C,\ d,\ p,\ l,\ L_0, \text{ and } D,$$

are represented graphically by combining the two triangles of Plane Sailing and Parallel Sailing, as in Fig. 6, in which

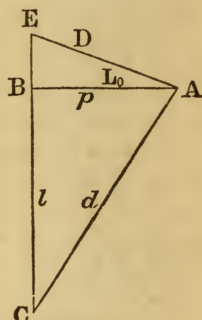


Fig. 6.

$$C = \text{A C B},$$
$$d = \text{C A},$$
$$p = \text{B A},$$
$$l = \text{C B},$$
$$L_0 = \text{B A E},$$
$$D = \text{A E}.$$

By these two right triangles, all the common cases classed under Middle Latitude Sailing (Bowd., p. 68) may be solved, if we add the formulas,

$$L' = L + l \qquad \lambda' = \lambda + D.$$

13. Other problems may be stated, which never occur in practice; as, for example,—

PROBLEM 6. *Given the course and distance, and the difference of longitude, to find both latitudes.*

Solution. We have, c, d, and D being given,

$$p = d \sin C,$$
$$l = d \cos C,$$
$$\cos L_0 = \frac{p}{D},$$
$$L = (L_0 - \varDelta L) - \tfrac{1}{2} l,$$
$$L' = (L_0 - \varDelta L) + \tfrac{1}{2} l,$$

$\varDelta L$ being taken from the table (Bowd., p. 76) corresponding to L_0.

Examples in Middle Latitude Sailing.

L and λ represent the latitude and longitude of the place sailed from.
L' and λ', the latitude and longitude of the place arrived at.

	L	λ	L'	λ'	COURSE.	DIST.
	° ′	° ′	° ′	° ′		′
1.	39 30 S.	74 20 E.	41 28 S	70 30 E.	S. W. by W.	210
2.	46 24 N.	47 15 W.	49 15 N.	42 21 W.	N. E. ½ E.	270
3.	51 10 S.	168 37 E.	48 31 S.	158 42 E.	....	...
4.	22 18 S.	57 28 E.	....	....	E. by S.	317
5.	23 15 S.	13 35 W.	....	....	E.	255
6.	20 5 N.	154 17 W.	18 28 S.	....	E. S E. ½ E.	...
7.	0 56 N.	29 34 W.	....	....	S. 47° E.	168
8.	45 16 S.	3 46 E.	43 10 S.	5 22 W.	....	...
9.	57 10 N.	178 51 W.	....	....	N. 6¾ pts. E.	290

10. Required the bearing and distance of Cape Race from Cape Hatteras.

$$\left[\tan C = \frac{D}{l} \cos (L_0 + \Delta L) \qquad d = l \sec C.\right]$$

	° ′	° ′	
Cape Hatteras,	35 15 N.	75 31 W.	Tab. LIV.
Cape Race,	46 39 "	53 5 "	
l	=11 24 "	= 684	
L_0	=40 57	D = 22 26 E. = 1346′	
ΔL	= + 17		
$L_0 + \Delta L$	=41 14		
log D	3.1290		
$l \cos (L_0 + \Delta L)$	9.8762		
ar co log l	7.1649	log l	2.8351
C=N. 55° 57′ E. $l.$ tan C	0.1701	l. sec C	0.2518
d =1222′		log d	3.0869

Note.—The logs of l and D may be obtained from the Table of "logarithms of small arcs in space or time" in the American Ephemeris and in Chauvenet's Lunar Method by regarding ′ and ″ as ° and ′.

11. A ship sails from Cape Frio south-easterly until her

departure is 3173 miles, and then, by observation, is in latitude 34° 30′ S.; required the course, distance, and longitude.

$$\left[\tan C = \frac{p}{l} \qquad d = l \sec C \qquad D = p \sec (L_0 + \varDelta L)\right]$$

Cape Frio,	23° 1′ S.	41° 59′ W.	
$L' =$	34 30 "	$p =$ 3173 E.	log p 3.5015
$l =$	11 29 "		log l 2.8382
$L_0 =$	28 45 "	$C =$ S. 77 45 E.	l. tan C 0.6633
$\varDelta L =$	17 "		l. sec C 0.6733
		$d =$ 3247	log c e 3.5115
$L_0 + \varDelta L =$	29 2 "	- - - - - -	l. sec 0.0583
		$D =$ 61 54 E.	log D 3.5698
		$\lambda' =$ 19 55 E.	

12. A ship in latitude 39° 8′ N., longitude 33° 45′ W., sails N. 51° 5′ E. 1014 miles; required her position.

$L' =$ 49° 45′ N. $\lambda' =$ 15° 16′ W.

13. A ship in latitude 56° 46′ S., longitude 170° 0′ E., sails E. N. E. until she is in latitude 50° 10′ S.; what is the distance sailed, and what is her longitude?

$d =$ 1035 miles, $\lambda' =$ 163° 9′ W.

14. A ship in 42° 42′ N., 12° 49′ W., sails 645 miles N. Wy., and is then in 49° 30′ N.; required the course and longitude in.

$C =$ N. 50° 46′ W. $\lambda' =$ 24° 51′ W.

15. A ship sails from Port Jackson in New-Holland N. 40° W. until the departure made is 300 miles; what is her position?

$L =$ 33° 50′ S. $L' =$ 27° 52′ S.
$\lambda =$ 151° 18′ E. $\lambda' =$ 145° 28′ E.

16. A ship in latitude 18° 50′ N., and longitude 153° 45′ W., sails S. E. ½ E., 3656 miles; what is her position? This example comes under Art. 11.

$$[l = d \cos C \quad D_1 = L \tan C \sec (\tfrac{1}{2} L + \varDelta L)$$
$$D_2 = L' \tan C \sec (\tfrac{1}{2} L' + \varDelta L')]$$

$C =$ S. $4\frac{1}{2}$ E.	l. cos C	9.8024	l. tan C 0.0858
$d = 3656'$	log d	3.5630	log L 3.0531
$l = 38° 40'$ S.	log l	3.3654	l. sec $(\frac{1}{2} L + \varDelta L)$ 0.0080
$L = 18° 50'$ N.	$D_1 =$	23° 22′ E.	log D_1 3.1469
$L' = 19° 50'$ S.			l. tan C 0.0858
$\frac{1}{2} L + \varDelta L =$	$9° 25' + 1° 31' =$	10° 56′ N.	log L' 3.0756
$\frac{1}{2} L' + \varDelta L' =$	$9° 55' + 1° 37' =$	11° 32′ S.	l. sec $(\frac{1}{2} L' + \varDelta L')$ 0.0089
$D_1 = 23° 22'$ E.	$D_2 =$	24° 40′ E.	log D_2 3.1703
	$D =$	48° 2′ E.	
	$\lambda =$	153° 45′ W.	
	$\lambda' =$	105° 43′ W.	

If $\varDelta L$ and $\varDelta L'$ are neglected, the resulting value of L' will be 105° 57′ W. If the computations are made with the middle latitude, 0° 30 S., λ' will be 106° 39′ W., or in error nearly 1°.

17. Find the latitudes of two places, whose longitudes are 12° 49′ W. and 24° 51′ W., their distance 645 miles, and the course from the first to the second N. 50° 46′ W. (Problem 6.)

$C =$	N. 50° 46′ W.		l. cos 9.8010	l. sin 9.8891
$d =$	645		log 9.8096	log 9.8096
$l =$	6 48 N.		log 9.6106	log p 9.6987
$D =$	12 2 W.		. . . . .	log 9.8585
$L_0 =$	46 12 N. or S.			l. cos 9.8402
$-\varDelta L =$	— 6			
	46 6 N.	or	46° 6′ S.	
$\frac{1}{2} l =$	3 24 N.		3 24 N.	
$L =$	42 42 N.	or	49 30 S.	
$L' =$	49 30 N.	or	42 42 S.	

Examples in Traverse Sailing.

A ship from the position given at the head of each of the following traverse forms sails the courses and distances

stated in the first two columns; required her latitude and longitude.

1. August 8, noon—Lat. by Obs., 35° 35′ N.
Long. by Chro. 18° 38′ W.

Courses.	Dist.	N.	S.	E.	W.
	′	′	′	′	′
N. N. E. ½ E.	50	44.1		23.6	
S. ¾ W.	46.2		45.7		6.7
S. by E. ½ E.	16.5		15.8	4.8	
N. E.	38	26.9		26.9	
S. S. W. ¼ W.	41.8		37.8		17.9
	192.5	71.0	99.3	55.3	24.6
S. 4¼ E.	41.5		28.3	30.7	
				38 =	D.

August 9, noon—Lat by Acct., 35° 7′ N.
Long. " 18° 0′ W.

2. September 25, noon—Lat. by Obs., 49° 53′ S.
Long. by Acct., 158° 27′ E.

Courses.	Dist.	N.	S.	E.	W.
Pts.	′	′	′	′	′
S. 4½ E.	45.3		28.7	35.0	
S. 5¼ E.	19.5		10.0	16.7	
S. 7 W.	38		7.4		37.3
S. 6¼ W.	25.7		8.7		24.2
S. 3 W.	51.2		42.6		28.4
N. 7¼ E.	13	1.9		12.9	
N. 5¾ E.	10	4.3		9.0	
	202.7	6.2	97.4	73.6	89.9
S. 1 W.	93		91.2		16.3
				D =	26

September 26, 8 A.M.—Lat. by Acct., 51° 24′ S.
Long. " 158° 1′ W.

In this example the courses are expressed in *points*, which is the preferable method.

When the reductions are the same for all the compass courses, we may find the difference of latitude and depar-

ture for these compass courses, and the course and distance made good. The traverse is thus referred to the magnetic meridian instead of the true. The course made good may then be corrected for variation, etc.; and with this corrected course and the distance made good the proper difference of latitude and departure may be found.

3. September 16, 6 P.M.—Lat by Obs., 50° 16′ S.
Long. by Chro., 76° 10′ W.

Comp. Course.	Dist.	N.	S.	E.	W.
	′	′	′	′	′
S. W. ½ S.	25		19.3		15.9
S. S. W.	30		27.7		11.5
S. by W.	18		17.7		3.5
S.	43		43		
S. by E. ¼ E.	25.5		24.7	6.2	
S. E. ¾ S.	33		26.5	19.7	
	174.5			25.9	30.9
(*mag.*) S. 2° W.	159		158.9		5.0
Var'n, &c., 18° W.					
(*true*) S. 16° E.	159		152.8	43.8	
or S. by E. ½ E.				70 =	D

September 17, noon—Lat. by Acct., 52° 49′ S.
Long. " 75° 0′ W.

MERCATOR'S SAILING.

14. Middle Latitude Sailing suffices for the common purposes of navigation; but a more rigorous solution of problems relating to the loxodromic curve is needed. These solutions come under "Mercator's Sailing."

PROBLEM 5. *A ship sails from the equator on a given course, C, till she arrives in a given latitude, L, to find the difference of longitude, D.*

Solution. Let the sphere (Fig. 7) be projected upon the plane of the equator stereographically. The primitive circle A B C....M is the equator.

P, its centre, is the pole (the eye or projecting point being at the other pole).*

The radii, P A, P B, P C, &c., are meridians making the same angle with each other in the projection as on the surface of the sphere.*

The distance P m, of any point m from the centre of the projection, $=\tan\frac{1}{2}(90°-L)$, the tangent of $\frac{1}{2}$ the polar distance of the point on the surface which m represents, the radius of the sphere being 1.*

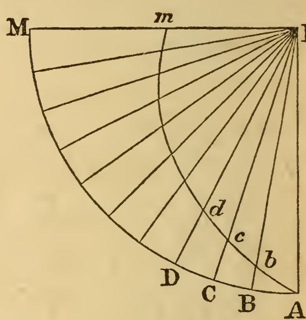


Fig. 7.

This curve in projection makes the same angle with each meridian, as the loxodromic curve with each meridian on the surface.*

A M is the whole difference of longitude D.

If we suppose this divided into an indefinite number of equal parts, A B, B C, C D, &c., each indefinitely small, and the meridians P A, P B, P C, &c., drawn, the intercepted small arcs of the curve A b c....m may be regarded as straight lines, making the angles P A b, P b c, P c d, &c., each equal to the course C; and consequently the triangles P A b, P b c, P c d, &c., similar.

We have then

$$\mathrm{PA} : \mathrm{P}b = \mathrm{P}b : \mathrm{P}c = \mathrm{P}c : \mathrm{P}d, \text{ \&c.,}$$

or the geometrical progression,

$$\mathrm{PA} : \mathrm{P}b : \mathrm{P}c : \ldots . \mathrm{P}m.$$

If then

$D =$ the whole difference of longitude,

$d =$ one of the equal parts of D,

$\dfrac{D}{d}$ will be the number of parts, and

$\dfrac{D}{d}+1$ the number of meridians P A, P b....P m,

---

* Principles of stereographic projection.

or the number of terms of the geometrical progression: and, employing the usual notation,

the first term $a = \text{P A} = 1$,

the last term $l = \text{P}\,m = \tan \frac{1}{2}(90° - L)$,

$$n - 1 = \frac{D}{d},$$

the ratio $$r = \frac{\text{P}\,b}{\text{P A}}.$$

To find this ratio, we have in the indefinitely small right triangle A B b,

$$\tan \text{B A}\,b = \cot \text{P A}\,b = \frac{\text{B}\,b}{\text{B A}},$$

or

$$\cot C = \frac{\text{P A} - \text{P}\,b}{d},$$

whence $$\text{P A} - \text{P}\,b = d \cot C;$$

$$\text{P}\,b = \text{P A} - d \cot C,$$

and, since P A=1,

$$r = \frac{\text{P}\,b}{\text{P A}} = 1 - d \cot C.$$

Then by the formula for a geometrical progression,

$$l = a\,r^{n-1},$$

(Algebra, p. 240,) we have

$$\tan \tfrac{1}{2}(90° - L) = (1 - d \cot C)^{\frac{D}{d}}. \qquad (8)$$

Taking the logarithm of each member, we have

$$\log \tan \tfrac{1}{2}(90° - L) = \frac{D}{d} \log (1 - d \cot C). \qquad (9)$$

But we have in the theory of logarithms

(*Naperian*) $$\log (1 + n) = n - \frac{n^2}{2} + \frac{n^3}{3} - \frac{n^4}{4} + \&c.....$$

and

(*Common*) $$\log (1 + n) = m\left[n - \frac{n^2}{2} + \frac{n^3}{3} - \frac{n^4}{4} + \&c...\right]. \qquad (10)$$

in which the modulus $m = .434294482$.

Hence, putting $n = -d \cot C$,

$$\log (1 - d \cot C) = m\left[-d \cot C - \tfrac{1}{2} d^2 \cot^2 C - \tfrac{1}{3} d^3 \cot^3 C - \&c....\right],$$

and substituting in (9) and reducing,

$$\log \tan \tfrac{1}{2}(90° - L) = -m \times D\left[\cot C + \tfrac{1}{2} d \cot^2 C + \tfrac{1}{3} d^3 \cot^3 C + \&c....\right]. \qquad (11)$$

This equation is the more accurate the smaller d is taken, so that if we pass to the limit and take $d=0$, it becomes perfectly exact. The broken line A $b\, c \ldots m$ then becomes a continuous curve, and our equation (11) becomes

$$\log \tan \tfrac{1}{2}(90°-L) = -m \times D \cot C;$$

whence

$$D = -\frac{\log \tan \frac{1}{2}(90°-L)}{m} \tan C, \tag{12}$$

But in this equation D is expressed in the same unit as $\tan C$, that is, in *terms of radius*. (Trig., Art. 11.)

To reduce it to minutes we must multiply it by the radius in minutes, or $r'=3437'.74677$.

Substituting the value of m, we shall have (in minutes),

$$D = -\frac{3437'.74677}{.434294482} \log \tan \tfrac{1}{2}(90°-L) \tan C.$$

To avoid the negative sign, we observe that

$$\tan \tfrac{1}{2}(90°-L) = \frac{1}{\cot \frac{1}{2}(90°-L)} = \frac{1}{\tan \frac{1}{2}(90°+L)},$$

or that

$$-\log \tan \tfrac{1}{2}(90°-L) = \log \tan \tfrac{1}{2}(90°+L).$$

Hence we have, by reducing,

$$D = 7915'.70447 \log \tan (45°+\tfrac{1}{2}L) \tan C. \tag{13}$$

NOTE.—Problem 5 may be more readily solved, and equation (13) obtained by aid of the Calculus.

In Fig. 1, suppose $c\,a$ to be an element, or infinitesimal part, of the loxodromic curve C A:

$c\,b$ will be the corresponding element of the meridian, and

$b\,a \times \sec L$, the element of the equator;

L being the latitude of the indefinitely small triangle $c\,a\,b$.

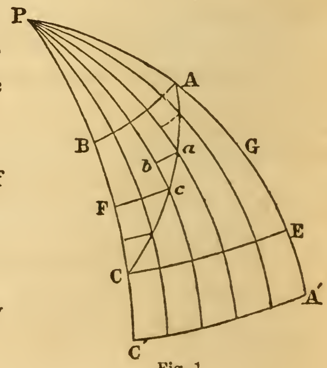


Fig. 1.

By articles 5 and 10, using the notations of the Calculus, we have

$$\mathrm{d}\,L = \cos C\,\mathrm{d}\,d \qquad \mathrm{d}\,p = \tan C\,\mathrm{d}\,L$$
$$\mathrm{d}\,D = \sec L\,\mathrm{d}\,p = \tan C \sec L\,\mathrm{d}\,L,$$

in which C is constant.

By integrating the last equation between the limits $L = 0$ and $L = L$, we shall have

$$D = \tan C \int_0^L \sec L\,\mathrm{d}\,L,$$

the whole difference of longitude required in Problem 5.

To effect the integration, put

$$\sin L = x, \qquad \text{then by differentiating}$$
$$\mathrm{d}\,L = \frac{\mathrm{d}\,x}{\cos L}, \qquad \text{and multiplying by sec } L$$
$$\sec L\,\mathrm{d}\,L = \frac{\mathrm{d}\,x}{\cos^2 L} = \frac{\mathrm{d}\,x}{1-\sin^2 L}, \qquad \text{or}$$
$$\sec L\,\mathrm{d}\,L = \frac{\mathrm{d}\,x}{1-x^2}.$$

Resolving into partial fractions, we obtain

$$\sec L\,\mathrm{d}\,L = \tfrac{1}{2}\left[\frac{\mathrm{d}\,x}{1+x} + \frac{\mathrm{d}\,x}{1-x}\right] \quad \text{and}$$
$$\int_0^L \sec L\,\mathrm{d}\,L = \tfrac{1}{2}\left[\log(1+x) - \log(1-x\right]$$
$$= \log\sqrt{\frac{1+x}{1-x}}$$
$$= \log\sqrt{\frac{1+\sin L}{1-\sin L}}$$
$$= \log\tan(45^\circ + \tfrac{1}{2} L) \qquad \text{Trig. (154).}$$

Whence we have

$$D = \log\tan(45^\circ + \tfrac{1}{2} L)\tan C.$$

But in this the logarithm is Naperian, and D is expressed in terms of the radius of the sphere. To reduce to common logarithms, we divide by $m = .434294482$, and to minutes by multiplying by $r' = 3437'.74677$, and obtain

$$D = 7915'.70447 \log\tan(45^\circ + \tfrac{1}{2} L)\tan C,$$

as in (13).

15. To facilitate the practical application of the formula just obtained, put

$$M = 7915'.70447 \log\tan(45^\circ + \tfrac{1}{2} L)\,; \qquad (14)$$

and let M be computed for each minute of L from 0 upward, and its values given in a table. We shall thus form the *Table of Meridional Parts* or of *Augmented Latitudes*, such as Bowditch's Table III. This formula accords with that given in the Preface. (Bowd., Pref. p. iv.)

In practice, then, we have only to take the value of M corresponding to L, and D is then found by the formula,

$$D = M \tan C. \tag{15}$$

M has the same name, or sign, as L.

Example.

To find tne meridional parts, or augmented latitudes, for each minute, from 30° to 33°;

$$\log 7915'.70447 = \underline{3.898490.}$$

$L.$	$45° + \frac{1}{2}L.$	*log tan.*	*l. log tan.*	*log M.*	$M.$		
° ′	° ′				′		
30 0	60 0	0.2385606	9.377599	3.276089	1888.37		
						23.14	
30 20	60 10	.2414830	.382887	.281377	1911.51		7
						23.21	
30 40	60 20	.2444154	.388129	.286619	1934.72		8
						23.29	
31 0	60 30	.2473580	.393326	.291816	1958.01		9
						23.38	
31 20	60 40	.2503108	.398480	.296969	1981.39		7
						23.45	
31 40	60 50	.2532741	.403591	.302081	2004.84		9
						23.54	
32 0	61 0	.2562480	.408660	.307150	2028.38		9
						23.63	
32 20	61 10	.2592328	.413690	.312180	2052.01		9
						23.72	
32 40	61 20	.2622286	.418680	.317170	2075.73		8
						23.80	
33 0	61 30	.2652356	.423632	.322122	2099.53		

The second differences afford a check of the work.

By interpolating into the middle, M can be found for each 10′; and then, by simple interpolation, for each 1′. In the first step, one eighth of the second difference is to be subtracted. The following is an example:

L	M	L	M
° ′	′	° ′	
30 0	1888.37	30 11	1901.09
1	1889.53	12	1902.24
2	1890.68	13	1903.40
3	1891.84	14	1904.56
4	1893.09	15	1905.72
5	1894.15	16	1906.87
6	1895.31	17	1908.03
7	1896.46	18	1909.19
8	1897.62	19	1910.35
9	1898.77	20	1911.51
10	1899.93	&c.	

16. PROBLEM 6. *A ship sails from a latitude, L, to another latitude, L', upon a given course, C; find the difference of longitude, D.*

Solution. Let

M be the augmented latitude corresponding to L,
M' " " " " L'.

The difference of longitude from the point, A, where the track crosses the equator to the 1st position, whose latitude is L, will be

$$D_{,}=M \tan C;$$

and to the second position, whose latitude is L',

$$D_{,,}=M' \tan C;$$

and we shall have

$$D=D_{,,}-D_{,}=(M'-M) \tan C; \tag{16}$$

or, when $M'<M$,

$$D=D_{,}-D_{,,}=(M-M') \tan C;$$

since the sign of D is determined by the course.

If L and L' are of different names, so also are M and M', and we have *numerically*

$$D=(M+M') \tan C.$$

17. The difference, $M'-M$, is called the *meridional*, or

augmented, difference of latitude. Representing this by m, we have

$$D = m \tan C. \tag{17}$$

The relation between these quantities is represented by a plane right triangle (Fig. 8), in which

C is one of the angles,
$m = \text{C E}$, the side adjacent,
$D = \text{E F}$, the side opposite.

The triangle of "Plane Sailing" has the same angle C, with

$l = \text{C B}$, the adjacent side,
and $p = \text{B A}$, the opposite side.

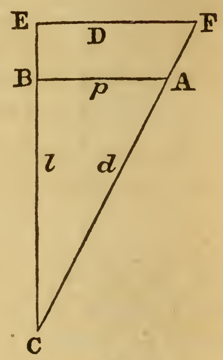


Fig. 8.

Fig. 8 represents these two triangles combined. By them, all the common cases under Mercator's Sailing can be solved, either by computation or by the Traverse Table. (Bowd., p. 79.)

The relations between the several parts involved are

$$\left.\begin{array}{ll} l = d \cos C, & L' = L + l, \\ p = d \sin C, & m = M' - M, \\ D = m \tan C, & \lambda' = \lambda + D; \\ \text{and since} \quad p = l \tan C, & \\ l : m = p : D. & \end{array}\right\} \tag{18}$$

18. Problem 7. *Given the latitudes and longitudes of two places, find the course, distance, and departure.* (Bowd., p. 79, Case I.)

Solution. L and L' being given, we take from Table III. M and M'.

We have $l = L' - L$, $m = M' - M$, $D = \lambda' - \lambda$;

by Mercator's sailing, $\tan C = \frac{D}{m}$;

and by Plane sailing, $d = l \sec C$, $p = l \tan C$;

l, m, and C are *north* or *south* according as L' is *north* or *south* of L.

D, p, and C are *east* or *west*, according as λ' is *east* or *west* of λ.

If the two places are on opposite sides of the equator, we have *numerically*

$$l = L' + L, \qquad m = M' + M.$$

Mercator's sailing is rarely used except in this case, and when the differences of latitude and longitude are considerable.

There are two limits of its accuracy:—

1. Table III. contains the augmented latitude only to the nearest *minute* or *mile*.*

2. It is computed on the supposition that the earth is a sphere. Some works on Navigation, as Mendoza Rios and Riddle, contain a table of augmented latitudes, in which the true form is taken into consideration.†

Examples.

1. Required the course and distance from Cape Frio to 34° 30′ S., 18° 30′ E.

	° ′		° ′		′		
Cape Frio,	23 1 S.		41 59 W.	$M =$	1420 S.		
$L' =$	34 30 S.	$\lambda' =$	18 30 E.	$M' =$	2208 S.	log D	3.5598
$l =$	11 29 S.	$D =$	60 29 E.	$m =$	788 S.	log m	2.8965
$C =$ S.	77 45 E.		l sec C		0.6733	l. tan C	0.6633
			log l		2.8382		
$d =$	3247′		log d		3.5115		

---

* The most convenient unit for nautical distances is the *geographical*, *nautical*, or *sea mile*, which is 1′ of the earth's equator, or 6086.43 feet. Regarding the earth as a sphere, this is also 1′ of any great circle.

† The formula for the terrestrial spheroid is

$$M = 7915'.70447 \log \tan (45° + \tfrac{1}{2} L)$$
$$-22'.98308 \sin L + 0'.01276 \sin 3L + \&c.$$

Delambre has shown that a table of meridional parts constructed for the sphere may be used for the spheroid by using as the argument the *geocentric* latitude instead of the true latitude.

2. Required the course and distance from Cape Frio to Lizard Point, England.

Cape Frio,	$23°\ 1'$ S.	$41°\ 59'$ W.	M $= 1420'$ S.	
Lizard Pt.,	49 58 N.	5 12 W.	$M' = 3471$ N.	log D 3.3438
$l =$	72 59 N.	$D =$ 36 47 E.	$M = 4891$ N.	log m 3.6894
$C =$ N. 24 17 E.		l. sec C	0.0402	l. tan C. 9.6544
		log l	3.6414	
$d =$	$4804'$	log d	3.6816	

3. A ship in latitude 18° 50′ N., longitude 153° 45′ W., sails S. 4½ points E., 3656 miles; what is her position?

		$d = 3656'$	log d 3.5630
$L = 18°\ 50'$ N.	$\lambda = 153°\ 45'$ W.	$M = 1151$ N.	l cos C 9.8024
$l = 38$ 40 S.			log l 3.3654
$L' = 19$ 50 S.		$M' = 1215$ S.	l. tan C 0.0858
		$M = 2366$ S.	log m 3.3740
	$D =$ 48 3 E.		log D 3.4598
	$\lambda' = 105$ 42 W.		

19. Other problems might be stated than those commonly given; as, for example,—

PROBLEM 8. *Given the latitude left, the course and both longitudes, to find the latitude arrived in.*

Solution. We have $D = \lambda' - \lambda$,

by Mercator's sailing $m = D \cot C$ (N. or S. as is C),

by Table III. M corresponding to L,

$M' = M + m$,

and again by Table III, L' corresponding to M'.

PROBLEM 9. *Given the difference of longitude and difference of latitude of two places, and the course between them, find both latitudes.*

Solution. We have

$$m = M' - M = D \cot C.$$

But

$$M' = 7915'.70447 \log \tan \tfrac{1}{2}\,(90° + L')$$
$$M = 7915'.70447 \log \tan \tfrac{1}{2}\,(90° + L),$$

consequently,

$$\log \tan \tfrac{1}{2}(90^\circ + L') - \log \tan \tfrac{1}{2}(90^\circ + L) = \frac{D \cot C}{7915'.70447}. \qquad (19)$$

Put
$$\log \cot \phi = \frac{D \cot C}{7915'.70447}, \qquad (20)$$

then equation (19) gives

$$\frac{\tan \frac{1}{2}(90^\circ + L')}{\tan \frac{1}{2}(90^\circ + L)} = \cot \phi.$$

By Pl. Trig. (109)

$$\frac{\tan \frac{1}{2}(x+y)}{\tan \frac{1}{2}(x-y)} = \frac{\sin x + \sin y}{\sin x - \sin y}.$$

In this, if we take

$$x + y = 90^\circ + L'$$
$$x - y = 90^\circ + L,$$

we have

$$x = 90^\circ + \tfrac{1}{2}(L' + L),$$

or putting

$$L_0 = \tfrac{1}{2}(L' + L) \text{ the middle latitude,}$$
$$x = 90^\circ + L_0,$$

and
$$y = \tfrac{1}{2}(L' - L) = \tfrac{1}{2} l,$$

and

$$\frac{\tan \frac{1}{2}(90^\circ + L')}{\tan \frac{1}{2}(90^\circ + L)} = \frac{\cos L_0 + \sin \frac{1}{2} l}{\cos L_0 - \sin \frac{1}{2} l} = \cot \phi,$$

whence

$$\cos L_0 = \frac{\cot \phi + 1}{\cot \phi - 1} \sin \tfrac{1}{2} l,$$

which, by Pl. Trig. (151), reduces to

$$\cos L_0 = \tan(45^\circ + \phi) \sin \tfrac{1}{2} l. \qquad (21)$$

We have also

$$\left.\begin{aligned} L &= L_0 - \tfrac{1}{2} l \\ L' &= L_0 + \tfrac{1}{2} l \end{aligned}\right\}. \qquad (22)$$

The solution is effected by equations (20), (21,) (22).

EXAMPLE.

The difference of longitude of two places is	5° 10′ E.
the difference of latitude,	3 28 N.
the course	N. 32 59 E;

find the latitudes.

(Constant) 7915′.70447	ar. co. log 6.10151
$D = 310'$	log 2.49136
$C = 32° 59'$	log cot 0.18776
$\log \cot \phi = 0.06034$	log 8.78063
$\phi = 41° 2'$	
$45° + \phi = 86° 2'$	log tan 1.15900
$\frac{1}{2} l = 1° 44'$	log sin 8.48069
$L_0 = 64° 8'$ N. or 64° 8′ S.	log cos 9.63969
$L = 62° 24'$ N. or 65° 52′ S.	
$L' = 65° 52'$ N. or 62° 24′ S.	

This problem cannot be solved with precision when L_0 is near 0.

20. PROBLEM 10. *To find the correction of the middle latitude in Mid. Lat. Sailing.* (Tab. Bowd., p. 76; Stanley, p. 338.)

Solution. In Mid. Lat. Sailing we have

$$\cos L_0 = \frac{p}{D}. \tag{23}$$

in which precision requires that we take

$$L_0 = \tfrac{1}{2}(L' + L) + \Delta L;$$

ΔL being a correction of the middle latitude, which it is now proposed to find.

In plane sailing $\quad p = l \tan C,$

in Mercator's sailing $\quad D = m \tan C,$

which substituted in (23) give

$$\cos L_0 = \frac{l}{m} \tag{24}$$

whence $$1-2\sin^2\tfrac{1}{2}L_0=\frac{l}{m},$$

and $$\sin\tfrac{1}{2}L_0=\sqrt{\frac{m-l}{2\,m}}. \tag{25}$$

Now for different values of L and L' we may find (in minutes) $l=L'-L$,
the middle latitude, $L_m=\frac{1}{2}(L'+L)$,
$m=7915'.70447\ [\log\tan(45°+\frac{1}{2}L')-\log\tan(45°+\frac{1}{2}L)]$,* and then L_0 by (24), or if small by (25), from which subtracting L_m we have $\varDelta L$, which is required.

In computing m, logarithms to 7 places should be used when the difference of latitude is less than 12°.

The correction of the Middle latitude computed for different *middle latitudes* and *differences of latitude* may be given in a table, as on page 76 (Bowd.) It becomes too large to be conveniently tabulated, when the latitudes are of different names, or the middle latitude is very small.

Example.

Find the correction of the Middle Latitude, when the latitudes are $L=12°$, $L'=18°$.

	° ′		
$(45°+\frac{1}{2}L')$	= 54 0	$l.$ tan 0.1387390	
$(45°+\frac{1}{2}L\,)$	= 51 0	$l.$ tan 0.0916308	const. log 3.89849
$\frac{1}{2}(L'+L)$	= 15 0	0.0471082	log 8.67310
l	= 6 0	= 360′	ar. co. log 7.44370
L_0	= 15 7		$l.$ sec 0.01529
$\varDelta L$	= 7		

21. The loxodromic curve on the surface of the earth and its stereographic projection (Fig. 7) present a peculiarity

* log 7915.70447 = 3.8984896. Another formula, requiring only 5 place logarithms, is
$$m=6875'.493\ \ [q+\tfrac{1}{3}q^3+\tfrac{1}{5}q^5+\tfrac{1}{7}q^7+\cdots]$$
in which $q=\sin\frac{1}{2}\,l\,\sec\frac{1}{2}(L'+L)$.

worthy of notice. Excepting a meridian and parallel of latitude, a line which makes the same angle with all the meridians which it crosses would continually approach the pole, until, after an indefinite number of revolutions, the distance of the spiral from the pole would become less than any assignable quantity. It is usual to say that such a curve meets the pole after an infinite number of revolutions. Still, however, it is limited in length.

For we have for the length of any portion,

by plane sailing, $d = (L' - L) \sec C.$

If $L = 0$ and $L' = 90° = \frac{\pi}{2}$,

the whole spiral from the equator to the pole will be, with radius $= 1$,

$$d = \frac{\pi}{2} \sec C.$$

If $L = -90° = -\frac{\pi}{2}$, and $L' = 90° = \frac{\pi}{2}$,

we have, as the entire length from pole to pole,

$$d = \pi \sec C.$$

If also $C = 0$, or the loxodromic curve is a meridian, $d = \pi$, a semicircumference, as it should be.

So also the length of the projected spiral $A\,b\,c\ldots$ (Fig. 7) from A to m can readily be shown to be (calling this length δ);

$$\delta = Mm \sec C = [1 - \tan(45° - \tfrac{1}{2}L)] \sec C,$$

or, (Pl. Trig. (151),) $\delta = \dfrac{2 \tan \frac{1}{2} L \sec C}{1 + \tan \frac{1}{2} L}$;

and its length from the equator to the pole—taking $L = 90°$;

$$\delta = \sec C.$$

A MERCATOR'S CHART.

22. On a Mercator's chart, the equator and parallels of latitude are represented by parallel straight lines; and the

meridians also by parallel straight lines at right angles with the equator. Two parallels of latitude, usually those which bound the chart, are divided into *equal parts*, commencing at some meridian and using some convenient scale to represent degrees, and subdivided to 10′, 2′, 1′, or some other convenient part of a degree, according to the scale employed.

Two meridians, usually the extremes, are also divided into degrees and subdivided like the parallels of latitude, but by a scale increasing constantly with the latitude: so that any degree of latitude on such meridian, instead of being equal to a degree of the equator, is the *augmented degree*, or augmented difference of 1° of latitude, derived from a table of "meridional parts." (Bowd., Table III.) The meridian is graduated most conveniently by laying off from the equator the *augmented latitudes;* or from some parallel, the *augmented difference of latitude* for each degree and part of a degree,—using the same scale of equal parts as for the equator.

Thus, on such a chart,

the length	of 1°	in lat.	0	is	60′	of the	equator,	
"	"	"	10°	"	61′	"	"	
"	"	"	20°	"	64′	"	"	
"	"	"	30°	"	69′	"	"	
"	"	"	40°	"	78′	"	"	
"	"	"	50°	"	93′	"	"	
"	"	"	60°	"	120′	"	"	
"	"	"	70°	"	176′	"	"	&c.,

and the augmented difference of latitude

from	0°	to	10°	is	603′ = 10° 3′	of	the	equator,	
"	10°	"	20°	"	622′ = 10° 22′	"		"	
"	20°	"	30°	"	663′ = 11° 3′	"		"	&c.

23. As on other maps and charts, parallels of latitude and

meridians are drawn at convenient intervals; places, shore lines of continents and islands, harbors and rivers, &c., are plotted, each point in its proper position; and such configurations of the land represented as the purpose of the map requires.

To plot on a chart a point, whose latitude and longitude are given,—by means of the scales at the sides, draw a parallel of latitude in the latitude, and by means of the scales at the top or bottom, a meridian in the longitude of the point; or so much of each as suffices to find their intersection. (Bowd., p. 88.)

In nautical charts the soundings in shoal water are put down, and even the character of the bottom; and on those of a large scale, also, the contour lines of the bottom, or lines of equal depth. The variation of the compass at convenient intervals, and lines of equal variation, are valuable additions.

24. The meridians on this chart being parallel, arcs of parallels of latitude are represented as equal to the corresponding arcs of the equator: thus each is expanded in the proportion of the secant of its latitude to 1; as is evident from the formula

$$D = p \sec L.$$

It can be shown that very small portions of the meridians are expanded in the same proportion. This is apparent from the table of the length of 1° in Art. 22; as for example, a degree whose middle latitude is 60° is 120′, or,

$$60' \text{ of the equator} \times \sec 60°.$$

But the two half degrees are unequally expanded; for

from $59\frac{1}{2}°$ to $60°$ is represented by $59'$,
" $60°$ to $60\frac{1}{2}°$ " " " $61'$, nearly.

A small circle on the surface of the earth of 1° diameter,

at the equator is then represented by a circle, whose diameter is 1°;

in lat.	30°	nearly by a circle,	whose	diameter	is	$1° \times \sec 30°$,
"	60°	"	"	"	"	$1° \times \sec 60°$,
"	L	"	"	"	"	$1° \times \sec L$;

but not exactly by a circle, since the meridians are augmented more rapidly as the latitude is greater.

Such a chart, then, while representing a narrow belt at the equator in proper proportions, presents a view of the earth's surface expanded at each point, both in latitude and longitude, proportionally to the secant of its latitude.

25. If we take any two points, C F, on this chart, and join them by a straight line, and form a right triangle by a meridian through one, and a parallel of latitude through the other, we shall have the triangle of Mercator's sailing (Fig. 8): for, the intercepted portion of the meridian, C E, is the augmented difference of latitude; and of the parallel of latitude, E F, is the difference of longitude. Hence the angle E C F is the course. (Art. 17.) Moreover, the loxodromic curve is represented by the straight line C F; for if we take any intermediate point of this curve, and let d be its position on the chart, d must be in the line C F, otherwise when we construct the triangle of Mercator's sailing, we shall have an angle at C different from E C F, the course; which for every point of the loxodromic curve is the same.

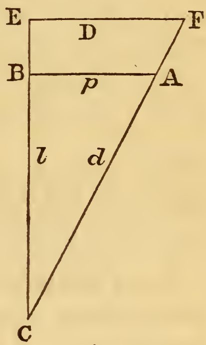


Fig. 8.

Thus a Mercator's chart presents two decided advantages for nautical purposes, viz.:

1. The ship's track is represented by a right line.
2. The angle, which this line makes with each meridian, is the course.

To find the course from one point to another on the chart, all that is necessary is to draw a line, or lay down the edge of a ruler, through the two points, and measure its angle with any meridian. A convenient mode is to refer such line by means of parallel rulers to the centre of one of the compass diagrams, which usually will be found on the chart, and reading the course from the diagram. Another mode of transferring the line to the compass diagram is described on page 88. (Bowd.)

As such diagrams, except on some charts of limited extent, are constructed with reference to the true meridian, the course obtained is the *true* course, and not the *compass* course.

26. The distance, C F, however, is an augmented distance, which we may measure nearly by the augmented scale on the meridians of the chart (the middle latitude of the scale used being the same as that of the line C E). (Bowd., p. 88.) Or we may construct the proper distance, C A, by constructing the triangle, C B A, of plane sailing, in which C B is the proper difference of latitude, the scale for which is on the equator.

The distance here spoken of, though represented on this chart by a straight line, is not the shortest distance between the two points,—for on the surface of a sphere, the shortest distance between two points is the arc of a great circle, which joins them. To find this belongs to *great-circle sailing.*

GREAT-CIRCLE SAILING.

27. The *rhumb-line*, or spiral curve, which cuts all the meridians at the same angle, has been used mostly by navigators in passing from point to point on account of the simplicity of the calculations required in practice. But, as has been stated, it is a longer line than the great circle between the same points, and therefore the intelligent navigators of

the present day are substituting the latter wherever practicable.

On the Mercator chart, however, the arc of a great circle joining two points, not on the equator or on the same meridian, will not be projected into a straight line, but into a curve longer than the Mercator distance, and still greater than the distance on a rhumb-line. Hence it is an objection to the Mercator chart, that the shortest route from point to point *appears* on it as a circuitous one; and this is, doubtless, one main reason why merely practical men have made so little use of the great circle. Many of those unacquainted with the mathematical principles of the subject are unable to comprehend how the apparently circuitous path on their chart should actually be the line of shortest distance.

28. PROBLEM 11. *To project on a chart the arc of a great circle joining two given points on the globe.*

Solution. It will be necessary to project a number of points of the arc, and trace through these points the curve by hand. To project a point on the chart, we must know its latitude and longitude.

The two given points, A and B (Fig. 9), and the pole, P, are the three angular points of a spherical triangle, formed by the arcs joining these points with each other and with the pole. If from P we draw P C_0 perpendicular to A B, the point C_0 is nearer the pole than any other point of A B; that is, it is the point of maximum latitude. This point of greatest latitude is called the *vertex* of the great circle.

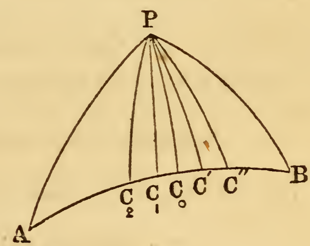


Fig. 9.

1st. *To find the latitude and longitude of this vertex.*

This may be done by a direct application of the rules of spherical trigonometry, first finding the angles A and B by

Case I. of Sph. Trig., and then solving one of the right triangles $A\ P\ C_0$ or $B\ P\ C_0$. But in practice the following method is preferable.

Let $L_1 = (90° - \mathrm{P\ A})$, the latitude of A, or the less latitude,
$L_2 = (90° - \mathrm{P\ B})$, the latitude of B, or the greater latitude,
$\lambda = \mathrm{A\ P\ B}$, the difference of longitude of B from A.
$L_v = (90° - \mathrm{P\ C_0})$, the latitude of the vertex.
$\lambda_1 = \mathrm{A\ P\ C_0}$, the longitude of the vertex from P A.
$\lambda_2 = \mathrm{B\ P\ C_0}$, the longitude of the vertex from P B.

The right triangle A P $\mathrm{C_0}$, gives

$$\cos \lambda_1 = \frac{\tan \mathrm{P\ C_0}}{\tan \mathrm{P\ A}} = \frac{\cot L_v}{\cot L_1} = \frac{\tan L_1}{\tan L_v}, \tag{26}$$

and the triangle B P $\mathrm{C_0}$,

$$\cos \lambda_2 = \frac{\tan \mathrm{P\ C_0}}{\tan \mathrm{P\ B}} = \frac{\cot L_v}{\cot L_2} = \frac{\tan L_2}{\tan L_v}.$$

Whence by division

$$\frac{\cos \lambda_1}{\cos \lambda_2} = \frac{\tan L_1}{\tan L_2};$$

and by composition and division,

$$\frac{\cos \lambda_2 - \cos \lambda_1}{\cos \lambda_2 + \cos \lambda_1} = \frac{\tan L_2 - \tan L_1}{\tan L_2 + \tan L_1}.$$

By Pl. Trig. (110) and (126), this equation becomes

$$\tan \tfrac{1}{2}(\lambda_1 + \lambda_2) \tan \tfrac{1}{2}(\lambda_1 - \lambda_2) = \frac{\sin (L_2 - L_1)}{\sin (L_2 + L_1)}.$$

But $\lambda = (\lambda_1 + \lambda_2)$, and if we put $\Delta \lambda = \tfrac{1}{2}(\lambda_1 - \lambda_2)$, we have

$$\tan \Delta \lambda = \frac{\sin (L_2 - L_1)}{\sin (L_2 + L_1)} \cot \tfrac{1}{2} \lambda, \tag{27}$$

$$\lambda_1 = \tfrac{1}{2} \lambda + \Delta \lambda, \tag{28}$$

$$\lambda_2 = \tfrac{1}{2} \lambda - \Delta \lambda. \tag{29}$$

By (27) we find $\Delta \lambda$, and then λ_1 by (28), which, applied to the longitude of A, or the point whose latitude is the smallest, and in the direction toward B, gives the longitude of the vertex.

For finding the latitude of the vertex, equations (26) give

$$\left.\begin{aligned} \tan L_v &= \tan L_1 \sec \lambda_1, \\ \tan L_v &= \tan L_2 \sec \lambda_2, \end{aligned}\right\} \qquad (30)$$

either of which may be used.

In using (27) attention must be paid to the signs of L_1 and L_2. If the greater latitude, L_2, is regarded as positive, L_1, when of a different name is negative; and in this case $(L_2 - L_1)$ will be numerically the sum, and $(L_2 + L_1)$ the difference of the two latitudes. In this case we shall find $\lambda_1 > 90°$.

When $\Delta\lambda > \frac{1}{2}\lambda$, λ_2 is negative, and the vertex, C_0, is beyond B, as in Fig. 10, instead of being between A and B, as in Fig. 9.

In (30) we have L_v positive, or of the same name with the greater latitude, since numerically $\lambda_2 < 90°$.

The vertex, which is here used, is that which is nearest the point B, or the place whose latitude is numerically the greater. For this in (27) $\Delta\lambda < 90°$.

There are, however, two vertices, which are diametrically opposite, as C_0 and C'_0 of the great circle $C'_0 E C_0$ in Fig. 10. For the vertex C', we have in (27) $\Delta\lambda > 180°$, or in the third quadrant, and in (30), L_v of a different name from L_2.

2d. *To find any number of points*, C', C'', C''', &c., C_1, C_2, C_3, &c., we may assume at pleasure the differences of longitude from the vertex $C_0 P C'$, $C_0 P C''$, $C_0 P C'''$, &c. It is best to assume them at equal intervals of 5° or 10°.

Let $\lambda' = C_0 P C'$, $\quad L' = (90° - P C')$, the lat. of C',
$\lambda'' = C_0 P C''$, $\quad L'' = (90° - P C'')$, " C'',
$\lambda''' = C_0 P C'''$, $\quad L''' = (90° - P C''')$, " C''',
&c. &c.

then the right triangles $C_0 P C'$, $C_0 P C''$, $C_0 P C'''$, &c., give

$$\left.\begin{aligned} \tan L' &= \tan L_v \cos \lambda', \\ \tan L'' &= \tan L_v \cos \lambda'', \\ \tan L''' &= \tan L_v \cos \lambda''', \text{ \&c.} \end{aligned}\right\} \qquad (31)$$

Or we may assume values of L', L'', L''', &c., and find the corresponding values of λ', λ'', λ''', &c., by the formulas

$$\left.\begin{aligned} \cos\lambda' &= \tan L' \cot L_v \\ \cos\lambda'' &= \tan L'' \cot L_v \\ \cos\lambda''' &= \tan L''' \cot L_v \text{ \&c.} \end{aligned}\right\} \qquad (32)$$

from which we shall have two values of λ for each value of L.

Having thus found as many points as may be deemed sufficient, we may plot them upon the chart, and through them trace the required curve.

29. Another method consists in finding the longitude of the point E (Fig. 10), where the great circle intersects the equator, and then by the right triangles E C′ c′, E C″ c″, E C‴ c‴, &c., the latitudes and longitudes of C′, C″, C‴, &c.

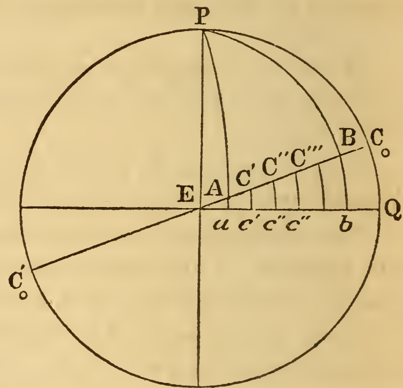


Fig. 10.

Let $\lambda_1 = \text{E}\,a$, the longitude of A from E;

$\lambda_2 = \text{E}\,b$, the longitude of B from E;

$\lambda = \lambda_2 - \lambda_1 = ab$, the difference of longitude of B from A;

L_1 and L_2, respectively the latitudes of A and B;

$L_v = \text{Q C}_0$, the latitude of the vertex, and also the measure of Q E C_0, the inclination of the great circle to the equator.

From the right triangles E A a, E B b, we have

$$\tan \text{E} = \frac{\tan \text{A}\,a}{\sin \text{E}\,a} = \frac{\tan \text{B}\,b}{\sin \text{E}\,b},$$

or

$$\tan L_v = \frac{\tan L_1}{\sin \lambda_1} = \frac{\tan L_2}{\sin \lambda_2};$$

whence
$$\frac{\sin \lambda_1}{\sin \lambda_2} = \frac{\tan L_1}{\tan L_2};$$

and by composition and division,

$$\frac{\sin \lambda_2 + \sin \lambda_1}{\sin \lambda_2 - \sin \lambda_1} = \frac{\tan L_2 + \tan L_1}{\tan L_2 - \tan L_1}.$$

By Pl. Trig., (109) and (126), this becomes

$$\frac{\tan \frac{1}{2} (\lambda_2 + \lambda_1)}{\tan \frac{1}{2} (\lambda_2 - \lambda_1)} = \frac{\sin (L_2 + L_1)}{\sin (L_2 - L_1)}.$$

But $\lambda = \lambda_2 - \lambda_1$, and if we put $\varDelta'\lambda = \frac{1}{2} (\lambda_2 + \lambda_1)$ we have

$$\left.\begin{aligned} \tan \varDelta'\lambda &= \frac{\sin (L_2 + L_1)}{\sin (L_2 - L_1)} \tan \tfrac{1}{2} \lambda \\ \lambda_1 &= \varDelta'\lambda - \tfrac{1}{2} \lambda \\ \lambda_2 &= \varDelta'\lambda + \tfrac{1}{2} \lambda \end{aligned}\right\} \qquad (33)$$

From these we find λ_1, or λ_2, which applied to the longitude of A, or of B, gives the longitude of E, the intersection with the equator. This point, it may be observed, is nearer A than B; and is outside of A, when the two places are on the same side of the equator; and between A and B when they are on different sides.

For finding the inclination of the circle to the equator, we have above

$$\left.\begin{aligned} \tan L_v &= \tan L_1 \operatorname{cosec} \lambda_1 \\ \tan L_v &= \tan L_2 \operatorname{cosec} \lambda_2 \end{aligned}\right\} \qquad (34)$$

in which L_v will have the same name, or sign, as L_2.

To find any numbers of points, C′, C″, C‴, &c., we may assume at pleasure, as in Art. 28, the differences of longitude, λ', λ'', λ''', &c., from E; and from the right triangles E C′ c', E C″ c'', E C‴ c''', &c., find the corresponding latitudes L', L'', L''', &c., by the formulas

$$\left.\begin{aligned} \tan L' &= \tan L_v \sin \lambda' \\ \tan L'' &= \tan L_v \sin \lambda'' \\ \tan L'' &= \tan L_v \sin \lambda''' \text{ \&c.} \end{aligned}\right\} \qquad (35)$$

The great circle intersects the equator at two opposite

points. The intersection, E, given by these formulas is that which is nearest A, the place whose latitude is the smallest.

This method is preferable to that of Art. 28, only when the two places are on different sides of the equator, and the intersection with the equator is between them. In this case, λ_1 and λ_2, as well as L_1 and L_2, will have opposite signs.

30. Problem 12. *To find the great-circle distance of two given points.*

Solution. Let A and B (Fig. 9) be the two given points.

In the triangle P A B are given, as in Problem 11,

$$\text{P A} = 90° - L_1, \qquad \text{P B} = 90° - L_2, \qquad \text{A P B} = \lambda,$$

to find $\text{A B} = d$, the distance required.

1*st Method.* By (27), (28), (29), and (30), we may find L_v, the latitude of the vertex, and λ_1 and λ_2, the longitudes of the two given places from the vertex.

Then from the right triangles, A C_0 P, B C_0 P, (Fig. 9), putting A $C_0 = d_1$ and B $C_0 = d_2$, we have

$$\left.\begin{aligned} \tan d_1 &= \tan \lambda_1 \cos L_v \\ \tan d_2 &= \tan \lambda_2 \cos L_v \\ \text{and} \qquad d &= d_1 + d_2. \end{aligned}\right\} \qquad (36)$$

d reduced to minutes will be the distance in geographical miles.

When λ_2 is negative, which happens when the vertex is beyond B (Fig. 10), d_2 is also negative, and d is numerically the difference of d_1 and d_2.

2*d Method.* By (33) and (34) we may find λ_1 and λ_2 the longitudes of the two given places from E (Fig. 10), the intersection, and L_v the angle which the circle makes with the equator.

Then from the right triangles, E a A, E b B, we have, putting $d' = \text{E A}$, $\quad d'' = \text{E B}$,

$$\left.\begin{aligned} \tan d' &= \tan \lambda_1 \sec L_v \\ \tan d'' &= \tan \lambda_2 \sec L_v \\ \text{and} \qquad d &= d'' - d'. \end{aligned}\right\} \qquad (37)$$

When L_1 and L_2 are of different names, so also are λ_1 and λ_2, and d is numerically the sum of d' and d''.

3*d Method.* By Sph. Trig. (4) we have

$$\cos d = \sin L_1 \sin L_2 + \cos L_1 \cos L_2 \cos \lambda,$$

and, putting

$$\left.\begin{array}{l} \left.\begin{array}{l} k \cos \phi = \sin L_2 \\ k \sin \phi = \cos L_2 \cos \lambda \end{array}\right\} \text{ or } \tan \phi = \cot L_2 \cos \lambda, \\ \cos d = \dfrac{\sin L_2 \sin (L_1 + \phi)}{\cos \phi}, \end{array}\right\} \quad (38)$$

from which ϕ and then d may be found. ϕ is in the same quadrant with λ.

30. Problem 13. *To find the course on a great-circle route.*

Solution. If A (Fig. 9), the point whose latitude is numerically the smaller, is the point of departure, it is required to find the angle P A B: if B is the point of departure, then the angle P B A.

1*st Method.* The position of the vertex having been found by (27), (28), (29), (30), we have from the right triangles, A C_0 P, B C_0 P,

$$\left.\begin{array}{l} \cos A = \sin L_v \sin \lambda_1 \\ \cos B = \sin L_v \sin \lambda_2 \end{array}\right\} \quad (39)$$

in which A $<$ 90°; and B $<$ 90°, when the vertex is between A and A (Fig. 9), but $>$ 90°, when the vertex is beyond B (Fig. 10).

2*d Method.* Having found the intersection and angle with the equator by (33) and (34), we have from the right triangles E a A, E b B (Fig. 10),

$$\left.\begin{array}{l} \cos A = \sin L_v \cos \lambda_1 \\ \cos B = -\sin L_v \cos \lambda_2 \end{array}\right\} \quad (40)$$

3*d Method.* By Napier's Analogies we have

$$\left.\begin{aligned}\tan\tfrac{1}{2}(B+A) &= \frac{\cos\tfrac{1}{2}(L_2-L_1)}{\sin\tfrac{1}{2}(L_2+L_1)}\cot\tfrac{1}{2}\lambda\\ \tan\tfrac{1}{2}(B-A) &= \frac{\sin\tfrac{1}{2}(L_2-L_1)}{\cos\tfrac{1}{2}(L_2+L_1)}\cot\tfrac{1}{2}\lambda\\ A &= \tfrac{1}{2}(B+A)-\tfrac{1}{2}(B-A)\\ B &= \tfrac{1}{2}(B+A)+\tfrac{1}{2}(B-A)\end{aligned}\right\}\qquad(41)$$

When A and B are on opposite sides of the equator, $\frac{1}{2}(L_2-L_1)$ is numerically half the sum, and $\frac{1}{2}(L_2+L_1)$ is half the difference of the two latitudes.

31. When the courses are found by this last method, the distance may be found by

$$\left.\begin{aligned}\tan\tfrac{1}{2}d &= \frac{\sin\tfrac{1}{2}(B+A)}{\sin\tfrac{1}{2}(B-A)}\tan\tfrac{1}{2}(L_2-L_1),\\ \text{or,}\quad \tan\tfrac{1}{2}d &= \frac{\cos\tfrac{1}{2}(B+A)}{\cos\tfrac{1}{2}(B-A)}\cot\tfrac{1}{2}(L_2+L_1),\end{aligned}\right\}\qquad(42)$$

The 1st is preferable, when $\frac{1}{2}(L_2+L_1)$ is near 0, and consequently $\frac{1}{2}(B+A)$ is near 90°; the second when $\frac{1}{2}(L_2-L_1)$ and consequently $\frac{1}{2}(B-A)$ are near 0. (Sph. Trig. 74.)

32. *Example.* To find the great circle from San Francisco to Jedo. (Formulas 27, 28, 29, 30.)

San Francisco,	Lat. L_2=	37° 48′ N.	Long 122° 22′ W.	
Jedo,	L_1=	35 40 N.	140 0 E.	
	L_2+L_1=	73 28	l. cosec 0.0183	
	L_2-L_1=	2 8	l. sin 8.5708	
	λ =	97 38		
	$\frac{1}{2}\lambda$ =	48 49	l. cot 9.9420	
	$\Delta\lambda$ =	1 57	l. tan 8.5311	
	$\lambda_1=\frac{1}{2}\lambda-\Delta\lambda$ =	50 46	l. sec λ_1 0.1990	l. sec λ_2 0.1651
	$\lambda_2=\frac{1}{2}\lambda-\Delta\lambda$ =	46 52	l. tan L_1 9.8559	l. tan L_2 9.8897
Vertex, L_v=48° 37′ N.	Long.	169 14 W.	l. tan L_v 0.0549	...... 0.0548

Long. from Vertex.	l. cos λ.	l. tan L.	Lat.	Longitudes.		
°			° ′	° ′	° ′	
0	0.0000	0.0549	48 37 N.	169 14 W.	169 14 W.	(Vertex).
± 5	9.9983	.0532	48 30	164 14	174 14	
±10	.9933	.0482	48 10	159 14	179 14 W.	
±15	.9849	.0398	47 37	154 14	175 46 E.	
±20	.9729	.0278	46 50	149 14	170 46	
±25	.9573	.0122	45 48	144 14	165 46	
±30	.9375	9.9924	44 30	139 14	160 46	
±35	.9134	.9683	42 55	134 14	155 46	
±40	.8842	.9391	41 0	129 14	150 46	
±45	.8494	.9043	38 44	124 14	145 46	
±50	.8081	.8630	36 7	119 14 W.	140 46 E.	

(36). l. tan λ_1 0.0880
l. tan λ_2 0.0283
l. cos L_v 9.8202
l. tan d_1 9.9082
l. tan d_2 9.8485

(39) l. sin λ_1 9.8891
l. sin λ_2 9.8632
l. sin L_v 9.8753
l. cos C_1 9.7444
l. cos C_2 9.7385

$d_1 = 39°\ 0'$ Course, C_1 N. 54° 27′ E. from Jedo.
$d_2 = 35\ \ 13$ C_2 N. 56 48 W. from San Francisco.

Distance, $d = 74°\ 13' = 4{,}453$ miles.
Distance by Mercator's sailing, 4,689 miles.

33. To follow a great circle rigorously requires a continual change of the course. As this is difficult, and indeed in many cases is practically impossible, on account of currents, adverse winds, &c., it is usual to sail from point to point by compass, thus making rhumb-lines between these points.

When the ship has deviated from the great circle which it was intended to pursue, it is necessary to make out a new one from the point reached to the place of destination. It is a waste of time to attempt to get back to an old line.

34. As the course, in order to follow a great circle, is practically the most important element to be determined, mechanical means of doing it have been devised. Towson's Chart and Table is used much by English navigators.

Chauvenet's Great-Circle Protractor renders it as easy as taking the rhumb-line course from a Mercator chart.

Charts have been constructed by a gnomonic projection, on which great circles are represented by straight lines; but by these computation is necessary to find the course.

35. A great circle between two points near the equator or near the same meridian differs little from a loxodromic curve. But when the differences both of latitude and of longitude are large, the divergence is very sensible. It is then that the great circle, as the line of shortest distance, is preferred.

But it is to be noted, that in either hemisphere the great-circle route lies nearer the pole, and passes into a higher latitude, than the loxodromic curve. Should it reach too high a latitude, it is usually recommended to follow it to the highest latitude to which it is prudent to go, then follow that parallel until it intersects the great circle again.

36. A knowledge of great-circle sailing will often enable the navigator to shape his course to better advantage. Let

A B (Fig. 11) be the loxodromic curve on a Mercator's chart, A C B, the projected arc of a great circle.

The length on the globe of the great circle A C B is less than that of the rhumb-line A B, or of any other line, as A D B, between the two. But A C B is also less than lines that may be drawn from A to B on the other side of it, that is, nearer the pole; and there will be some line, as A D′ B, nearer the pole than the great circle, and equal in length to the rhumb-line. Between this and the rhumb-line may be drawn curves from A to B, all less than the rhumb-line. If the wind should prevent the ship from sail-

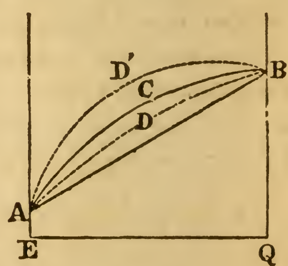


Fig. 11.

ing on the great circle, a course as near it as practicable should be selected. If she cannot sail between A B and A C, there is the choice of sailing nearer the equator than A B, or nearer the pole than A C. The ship may be nearing the place B better by the second than by the first, although on the chart it would appear to be very far off from the direct course.

This may be strikingly illustrated by the extreme case of a ship from a point in a high latitude to another on the same parallel 180° distant in longitude. The great-circle route is across the pole, while the rhumb-line is along the small circle, the parallel of latitude, east or west; the two courses differing 90°. Any arc of a small circle drawn between the two points, and lying between the pole and the parallel of latitude, will be less than the arc of the parallel. Hence the ship may sail on one of these small circles nearly west, and make a less distance than on the Mercator rhumb, or parallel due east. This is, indeed, an impossible case in practice, but it gives an idea of the advantage to be gained in any case by a knowledge of the great-circle route.

It is possible in high latitudes that a ship may have such a wind as to sail close-hauled *on one tack* on the rhumb-line, and yet be approaching her port better by sailing on the *other tack*, or twelve points from the rhumb-line course.

37. The routes between a number of prominent ports recommended by Captain Maury are mainly great-circle routes, modified in some cases by his conclusions respecting the prevailing winds.

SHAPING THE COURSE.

38. The intelligent navigator, in selecting his course to a destined port, will not only have regard to the directness of the route, but will take into consideration obstructions and dangers which may be in his way; prevailing winds

and currents; and, in case of a threatening storm, the course to be taken to avoid its greatest violence, or being driven on a lee shore.

Good charts and books of sailing directions afford all requisite information respecting obstructions and dangers in the most frequented seas. Exploring expeditions from England, France, and the United States have of late years added greatly to this branch of nautical knowledge.

The labors of Maury, and his recent colaborers in England and France, have greatly increased our knowledge of prevailing winds in large portions of the ocean. The careful observations of intelligent navigators are much needed still further to develop it.

A few of the stronger currents, such as the Gulf Stream off the coast of the United States, are well known. But more extended observations are wanted. Currents are often indicated by the difference in the temperature of their waters from that of those surrounding them; so that the thermometer, as well as barometer, has become an important instrument to the navigator.

The works of Redfield, and especially of Reid and Piddington, afford much information respecting storms and tornadoes. That class of storms called cyclones is particularly deserving of attention.

These branches of physical geography are well worthy of study by those engaged in navigating the ocean.

CHAPTER II.

REFRACTION.—DIP OF THE HORIZON.—PARALLAX.—SEMIDIAMETERS.

REFRACTION.

39. It is a fundamental law of optics, that a ray of light passing from one medium into another of different density is refracted, or bent, from a rectilinear course. If it passes from a lighter to a denser medium, it is bent toward the perpendicular to the surface, which separates the two media; if it passes from a denser to a lighter medium, it is bent from that perpendicular. Let

M and N (Fig. 12) represent two media each of uniform density, but the density, or refracting power, of N being the greater;

a b c, the path of the ray of light through them;

P *b*, the normal line, or perpendicular, to the separating surface at *b*.

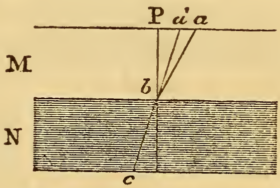


Fig. 12.

If *a b* is the *incident ray*, *b c* is the *refracted ray;* P *b a* is the *angle of incidence;* P *b a'* is the angle of *refraction.*

If *c b* is the incident ray, *b a* is the refracted ray, and P *b a'* and P *b a* are respectively the angles of incidence and refraction.

Moreover, these angles are in the same plane, which, as it passes through P *b*, is perpendicular to the surface at which the refraction takes place; and we have for the *refraction*

$$a' \, b \, a = \mathrm{P} \, b \, a - \mathrm{P} \, b \, a',$$

or *the difference of direction of the incident and refracted rays.*

A more complete statement of the law for the same two media is, that

$$\frac{\sin \mathrm{P} \, b \, a}{\sin \mathrm{P} \, b \, a'} = m, \text{ a constant for these media;}$$

or, *the sines of the angles of incidence and refraction are in a constant ratio.*

This law is also true when the surface is curved as well as when it is a plane.

40. If the medium N instead of being of uniform density is composed of parallel strata, each uniform but varying from each other, the refracted ray $b\,c$ will be a broken line; and if, as in Fig. 13, the thickness of these strata is indefinitely small, and the density gradually increases in proceeding from the surface b, $b\,c$ will become a curved line. But we shall still have for any point c of this curve, $c\,a'$ being a tangent to it,

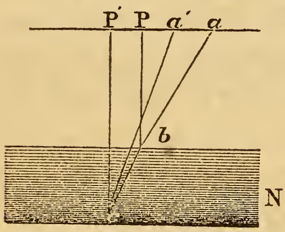


Fig. 13.

$$\frac{\sin \mathrm{P} \, b \, a}{\sin \mathrm{P}' \, c \, a'} = m,$$

a constant for the particular strata in which c is situated.

This law, which is true for strata in parallel planes, extends also to parallel spherical strata, except that the normals P b, P′ c are no longer parallel, but will meet at the centre of the sphere. But the refraction takes place in the common plane of these two normals.

41. The earth's atmosphere presents such a series of parallel spherical strata, denser at the surface of the earth, and decreasing in density, until at the height of fifty miles the refracting power is inappreciable.

In Fig. 14, the concentric circles M N represent sections of these parallel strata, formed by the vertical plane passing through the star S and the zenith of an observer at A. The normals C A Z at A, and C B E at B, are in this vertical plane. S B, a ray of light from the star S, passes through the atmosphere in the curve B A, and is received by the observer at A.

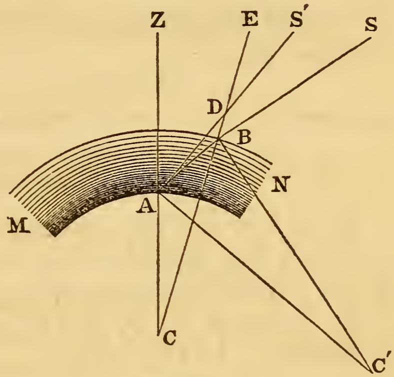


Fig. 14.

Let A S′ be a tangent to this curve at A; then the *apparent* direction of the star is that of the line A S′; and the astronomical refraction is the difference of directions of the two lines B S and A S′. This difference of directions is the difference of the angles E B S, E D S′, which the lines S B, S′ A make with any right line C B E, which intersects them. If, then, r represent the refraction, we have

$$r = \text{E B S} - \text{E D S}'.$$

Also, E B S is the angle of incidence, and Z A S′, the apparent zenith distance, is the angle of refraction; and we have

$$\frac{\sin \text{E B S}}{\sin \text{Z A S}'} = m,$$

a constant ratio for a given condition of the atmosphere and a given position of A; but varying with the density of the atmosphere, and for different elevations of A above the surface. For a mean state of the atmosphere and at the surface of the earth, experiments give $m = 1.000294$.

The principles of Arts. 39 and 40, applied to this case,

show that astronomical refraction takes place in vertical planes, so as to increase the altitude of each star without affecting its azimuth. The refraction must therefore be subtracted from an observed altitude to reduce it to a true altitude; or

$$h = h' - r,$$

in which h is the true altitude,
h', the apparent altitude,
r, the refraction.

These laws are here assumed. The facts and reasoning on which they depend belong to works on optics. (Bowd., p. 153; Herschel's Astronomy, p. 37; Lardner's Optics.)

42. PROBLEM 14. *To find the refraction of a star.*

Solution. In Fig. 14, let

$z = \text{Z A S}'$, the apparent zenith distance of the star,
$r = \text{E B S} - \text{E D S}'$, the refraction,
$u = \text{Z C E}$;

then $\text{E D S}' = \text{A D C} = \text{Z A S}' - \text{Z C E} = z - u,$

$$\text{E B S} = \text{E D S}' + r = z - u + r,$$

and

$$\frac{\sin \text{E B S}}{\sin \text{Z A S}'} = \frac{\sin (z - u + r)}{\sin z} = m,$$

or

$$\sin [z - (u - r)] = m \sin z; \qquad (43)$$

which is of the same form as (309) of Plane Trig.,

$$\sin (z + a) = m \sin z,$$

the solution of which gives

$$\tan (z + \tfrac{1}{2} a) = \frac{m + 1}{m - 1} \tan \tfrac{1}{2} a.$$

Putting $a = -(u - r)$, its value in (43), we have

$$\tan [z - \tfrac{1}{2} (u - r)] = \frac{1 + m}{1 - m} \tan \tfrac{1}{2} (u - r);$$

whence

$$\tan \tfrac{1}{2} (u - r) = \frac{1 - m}{1 + m} \tan [z - \tfrac{1}{2} (u - r)]. \qquad (44)$$

In this u and r are both unknown, but we may note that each is a very small angle, being 0 when the zenith distance is 0, and increasing with the zenith distance. As it is necessary to make some supposition respecting them, let us assume that they vary proportionally, and that

$$\frac{u}{r} = q,$$

a constant, reserving it for observations to test the rigor of this assumption.

Equation (44) then becomes

$$\tan \tfrac{1}{2} (q-1)\, r = \frac{1-m}{1+m} \tan \left[z - \tfrac{1}{2} (q-1)\, r\right],$$

or, since $\frac{1}{2} (q-1)\, r$ is quite small,

$$\tfrac{1}{2} (q-1)\, r \sin 1'' = \frac{1-m}{1+m} \tan \left[z - \tfrac{1}{2} (q-1)\, r\right];$$

whence

$$r = \frac{2}{(q-1) \sin 1''} \cdot \frac{1-m}{1+m} \tan \left(z - \tfrac{1}{2} (q-1)\, r\right).$$

Since observation is to determine q, we may as well consider that it determines the whole of the constant factor into which q enters.

Put then $$n = \frac{2}{(q-1) \sin 1''} \cdot \frac{1-m}{1+m},$$
$$p = \tfrac{1}{2} (q-1),$$

and the formula reduces to

$$r = n \tan (z - p\, r),$$

which is known as Bradley's formula.

Suppose at two given zenith distances z' and z'' the refractions r' and r'' are found by observations in a mean state of the atmosphere, then we have the two equations,

$$r' = n \tan (z' - p\, r'),$$
$$r'' = n \tan (z'' - p\, r'');$$

and the two unknown quantities n and p may be found by proper transformations, or by successive approximations.

By comparing pairs of observations in this way at various zenith distances, the values of n and p come out *very nearly* the same, except at very low altitudes; so that the hypothesis that q, and therefore n and p are constant, is found to be nearly, though not rigorously exact.

The values that have been found are, with the barometer at 29.6 inches, and the thermometer at 50°,

$$n = 57''.036, \qquad p = 3;$$

and the formula by which Tab. XII. (Bowd.) has been computed is

$$r = 57''.036 \tan (z - 3\,r). \tag{46}$$

In computing by this formula, we must find an approximate value of r, by assuming first

$$r = 57''.036 \tan z,$$

and substitute the value thus obtained in the second member of the proper equation.

Example.

Find the refraction for the altitude 30°.

$\log 57''.036 = 1.75615$		. . . .	1.75615
$z = 60°$	l. tan 0.23856	$z - 3\,r = 59°55'4''$	l. tan 0.23712
$r = \quad 98''.8$	log 1.99471	$r = 98''.46$	log 1.99327
$3\,r = 4'\ 56''.4$		$r = 1'38''.46$	

43. Laplace, from a more profound investigation of this problem, obtained a more complicated formula, which agrees better with observations.

Bessel has modified and improved Laplace's formula. His tables of refractions are now considered the most reliable. They are found in a convenient form for nautical problems in Chauvenet's Method of Equal Altitudes, Table III. The mean refractions in this Table are for the height of the

barometer, 30 inches, and the temperature 50° of Fahrenheit.*

44. The mean values of n and p in Art. 42 correspond to

the height of the barometer, $b = 29.6$ inches,
the thermometer, $t = 50°$ Fahrenheit.

Now, the refraction in different conditions of the atmosphere is nearly proportional to the density of the air; and this density, the temperature being the constant, is proportional to its elasticity; that is, to the height of the barometer. Hence, if

b is the noted height of the barometer (in inches),

r, the mean refraction of Tab. XII.,

Δr, the correction for the barometer,

then
$$\frac{r + \Delta r}{r} = \frac{b}{29.6},$$

or
$$\Delta r = \frac{b - 29.6}{29.6} r. \tag{47}$$

By this formula the correction for the barometer in Tab. XXXVI. is computed.

Again, the elastic force being constant, the density increases by $\frac{1}{400}$ part for each depression of 1° Fahrenheit. Hence, if

$\Delta' r =$ the correction for the thermometer,

$t =$ the temperature in degrees of Fahrenheit,

$$\Delta' r = \frac{50° - t}{400°} (r + \Delta' r) \tag{48}$$

which reduces to $\Delta' r = \dfrac{50° - t}{350° + t} r$

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* Chauvenet's Astronomy, Vol. I. pp. 127–172, contains a thorough investigation of the problem of refraction, especially of Bessel's formulas.

by which the correction for the thermometer in Table XXXVI. is computed.

Bessel's formulas are more rigid, but more complex.

45. Problem 15. *To find the radius of curvature of the path of a ray in the earth's atmosphere.*

Solution. By the radius of curvature for any point of a curve, is meant the radius of the circular arc, which most nearly coincides with the curve at that point.

If we consider the curvature of the path of a ray to be uniform from B to A (Fig. 14), it is the same as considering the curve B A itself to be a circular arc, and the problem is reduced to finding the radius of this arc.

Let C′ be the centre of the arc A B,

$R' = $ C′ A, the radius of curvature,

$R = $ C A, the radius of the earth.

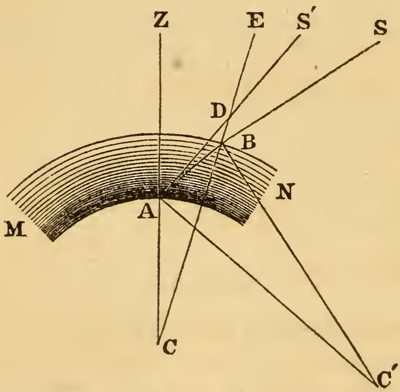

Fig. 14.

Since S B and S′ A are tangents to the curve at B and A respectively, they are perpendicular respectively to the radii C′ B, C′ A; hence,

$$\text{A C}'\text{ B} = r,$$

the difference of directions of S B and S′ A.

As A B is a very small arc, we may put

$$\text{A B} = R' \sin r;$$

and, since they are very nearly equal, we may also put

$$\text{A D} = \text{A B} = R' \sin r.$$

In the triangle A D C,

$$\frac{\text{A D}}{\text{A C}} = \frac{\sin \text{A C D}}{\sin \text{A D C}},$$

or
$$\frac{R' \sin r}{R} = \frac{\sin u}{\sin (z - u)};$$

whence
$$R' = \frac{R}{\sin (z - u)} \cdot \frac{\sin u}{\sin r}, \tag{49}$$

or nearly enough, since u and r are small,

$$R' = \frac{R}{\sin z} \cdot \frac{u}{r}. \tag{50}$$

But in the preceding problem

$$u = q\, r, \qquad p = \tfrac{1}{2}(q - 1) = 3,$$

whence
$$q = 7, \qquad u = 7\, r;$$

so that
$$R' = \frac{7\, R}{\sin z}, \tag{51}$$

which is the required formula, nearly.

46. When $z = 0$, or the star is in the zenith,

$$R' = \infty;$$

that is, the path is a straight line.

When $z = 90°$, or the object is in the horizon,

$$R' = 7\, R;$$

that is, near the earth's surface a ray of light nearly horizontal moves in a curve, which is nearly the arc of a circle whose radius is seven times the radius of the earth.

This, however, is in a mean condition of the atmosphere. The curve is greatly varied in extraordinary states of the atmosphere, or by passing near the earth's surface of different temperatures; in very rare cases even to the extent of becoming convex to the surface a short distance.

DIP OF THE HORIZON.

47. PROBLEM 16. *To find the dip of the horizon.*

Solution. Let A (Fig. 15) be the position of the observer at the height B A $= h$, above the level of the sea; A H, perpendicular to the vertical line, C A, represents the true horizon.

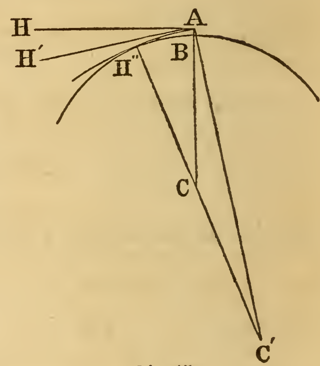

Fig. 15.

The most distant point of the horizon visible from A is that at which the visual ray, H″ A, is tangent to the earth's surface.

The apparent direction of H″ is A H′, the tangent to the curve A H″ at A. $\varDelta$ H = H A H′ is the dip of the horizon to be found.

Let C be the centre of the earth,
C′, the centre of the arc H″ A.

H″, C, C′, are in the same straight line, since the arcs H″ B, H″ A are tangent to each other at H″,

C A, C′ A, are perpendicular respectively to AH, A H′; hence

$$C\,A\,C' = H\,A\,H' = \varDelta\,H, \text{ the dip.}$$

Let $R = $ C B, the radius of the earth;

then $R + h = $ C A,

$7\,R = $ C′ A $=$ C′ H″, the radius of curvature of H″A,

$6\,R = $ C C′.

We have, then, in the triangle C A C′, by Pl. Trig. (268),

$$\sin \tfrac{1}{2}\,\varDelta\,H = \sqrt{\frac{(6\,R - \frac{1}{2}\,h)\,(\frac{1}{2}\,h)}{7\,R.(R+h)}};$$

and, since h is comparatively very small and may therefore be omitted alongside of R,

$$\sin \tfrac{1}{2} \varDelta \mathrm{H} = \sqrt{\frac{3h}{7R}};$$

or, putting $\sin \frac{1}{2} \varDelta H = \frac{1}{2} \varDelta H \sin 1''$,

$$\varDelta H = \frac{2}{\sin 1''} \sqrt{\frac{3h}{7R}} = \frac{2}{\sin 1''} \sqrt{\frac{3}{7R}} \sqrt{h}. \qquad (52)$$

Taking $R = 20923596$ feet (Herschel, p. 126), we find the constant factor

$$\frac{2}{\sin 1''} \sqrt{\frac{3}{7R}} = 59''.040,$$

$$\varDelta H = 59''.040 \sqrt{h}, \qquad (53)$$

and $\log \varDelta H = 1.77115 + \frac{1}{2} \log h$,

h being expressed in feet, which is nearly the formula for Tab. XIII. (Bowd.)

Since $\dfrac{2}{\sin 1''} \sqrt{\dfrac{3}{7R}}$ is constant, depending only upon the radius of the earth, $\varDelta H$ is proportional to $\sqrt{h}$, or the dip is proportional to the square root of the height of the observer above the level of the sea.

48. Were the path of the ray, H A, a straight line, we should have

$$\varDelta' H = \mathrm{H\,A\,H''} = \mathrm{H''\,C\,A},$$

and in the triangle H″ C A

$$\cos \varDelta' H = \frac{R}{R+h},$$

whence, $2 \sin^2 \frac{1}{2} \varDelta' H = \dfrac{h}{R+h} = \dfrac{h}{R}$, nearly,

and $\varDelta' H = \dfrac{2}{\sin 1''} \sqrt{\dfrac{h}{2R}}$,

or with h in feet, $\varDelta' H = 63''.771 \sqrt{h}. \qquad (54)$

Comparing this with $\varDelta H = 59''.040 \sqrt{h}$, we find

$$\varDelta H = \varDelta' H - 4''.731 \sqrt{h} = \varDelta' H - .074\ \varDelta' H,$$

or that the dip is decreased by refraction by .074, or nearly $\frac{1}{13}$ of it.

But from the irregularity of the refraction of horizontal rays (Art. 46), the dip varies considerably, so that the tabulated dip for the height of 16 feet can be relied on ordinarily only within 2′. When the temperatures of the air and water differ greatly, variations of the dip from its mean value as great as 4′ may be experienced. In some rare cases, variations of 8′ have been found.

The dip may be directly measured by a dip-sector. A series of such measurements carefully made, and under different circumstances, both as to the height of the eye, temperature and pressure of the atmosphere, and temperature of the water, are greatly needed.

Prof. Chauvenet (Astron. I., p. 176) has deduced the following formula, which it is desirable to test by observations:

$$\text{in seconds, } \Delta H = \Delta' H - 24021'' \frac{t - t_0}{\Delta' H},$$

or

$$\text{in minutes, } \Delta H = \Delta' H - 6'.67 \frac{t - t_0}{\Delta' H};$$

in which

t is the temperature of the air,
t_0 that of the water,

by a Fahrenheit thermometer.

When the sea is warmer than the air, the visible horizon is found to be below its mean position, or the dip is greater than the tabulated value; when the sea is colder than the air, the dip is less than its tabulated value. (Raper's Nav., p. 61.)

This uncertainty of the dip affects to the same extent all altitudes observed with the sea horizon.

49. Near the shore, or in a harbor, the horizon may be obstructed by the land. (Bowd., p. 155.) The shore-line may then be used for altitudes instead of the proper horizon. Tab.

XVI. (Bowd.) contains the dip of such water-line, or of any object on the water, for different heights in feet and distances in sea miles. It is computed by the formula

$$D = \frac{3}{7} d + 0.56514 \frac{h}{d} \tag{55}$$

in which

h is the height in feet;
d, the distance of the object in sea miles;
D, the dip in minutes.

50. Problem 17. *To find the distance of an object of known height, which is just visible in the horizon.*

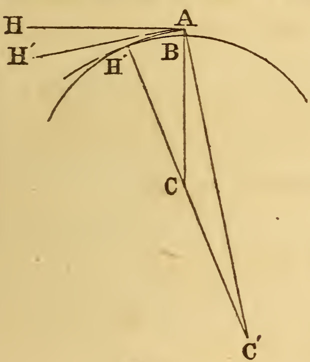

Fig. 15.

Solution. If the observer is at the surface of the earth at the point H″ (Fig. 15), a point A appears in the horizon, or is just visible, when the visual ray A H″ just touches the earth at H″. Let

$h = $ B A, the height of A,
$d = $ H″ A, the distance of A.

As this arc is very small, we have

$$d = \text{H}'' \text{ C}' \text{ A} \sin 1'' \times \text{C}' \text{ A} = 7\, R \times \text{H}'' \text{ C}' \text{ A} \sin 1'',$$

since by (51) $\qquad \text{C}' \text{ A} = 7\, R.$

From the three sides of the triangle C C′ A by Pl. Trig. (268),

$$\sin \tfrac{1}{2} \text{ H}'' \text{ C}' \text{ A} = \sqrt{\frac{\frac{1}{2} h\, (R + \frac{1}{2} h)}{42\, R^2}},$$

or nearly $\qquad \frac{1}{2} \text{ H}'' \text{ C}' \text{ A} \sin 1'' = \sqrt{\frac{h}{84\, R}},$

and $\qquad \text{H}'' \text{ C}' \text{ A} \sin 1'' = \sqrt{\frac{h}{21\, R}}.$

This, substituted in the expression for d, gives

$$d = 7\,R\sqrt{\frac{h}{21\,R}} = \sqrt{\left(\frac{7}{3}\,R\,h\right)}. \qquad (56)$$

In this, d, h, and R are expressed in the same denomination.

But if h and R are in feet,

in statute miles, $$d = \frac{1}{5280}\sqrt{\left(\frac{7}{3}\,R\,h\right)},$$

in geographical miles, $$d = \frac{1}{6087}\sqrt{\left(\frac{7}{3}\,R\,h\right)}.$$

Taking $R = 20923596$ feet as before, we find

$$\left.\begin{array}{ll} \text{in stat. miles } d = 1.323\ \sqrt{h}, & \text{or } \log d = 0.12172 + \tfrac{1}{2}\log h, \\ \text{in geog. `` } d = 1.148\ \sqrt{h}, & \text{or } \log d = 0.05994 + \tfrac{1}{2}\log h. \end{array}\right\} \quad (57)$$

The first of these is nearly the formula given by Bowditch for computing Table X. (Bowd., Preface.)

51. Were the visual ray, H″ A, a straight line, we should have from the right triangle $C\,H''\,A$,

$$\mathrm{H''\,A} = \sqrt{(\mathrm{C\,A}^2 - \mathrm{H''\,C}^2)}, \quad \text{or} \quad d' = \sqrt{(2\,R + h)\,h}\,;$$

or nearly $$d' = \sqrt{2\,R} \times \sqrt{h}.$$

Introducing the same numerical values as before, we have in statute miles

$$d' = 1.225\ \sqrt{h}.$$

Comparing this with the expression above, we see that the distance is increased about $\frac{1}{12}$ part by refraction. This, however, is subject to great uncertainty.

52. If the observer is also elevated at the height of B′ A′ (Fig. 16), and sees the object A in his horizon, then its distance is

$$\mathrm{A'\,H''} + \mathrm{H''\,A},$$

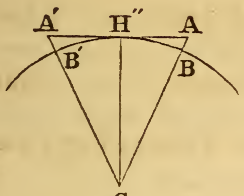

Fig. 16.

or the sum of the distances of each from the common horizon, H″.

By entering Table X. with the heights of the observer and the object respectively, the sum of the corresponding distances is the distance of the object from the observer. The distances in this table are in statute miles. Multiplying them by $\frac{5280}{6086.4} = .86751$, reduces them to geographical miles.

PARALLAX.

53. The change of the direction of an object, arising from a change of the point from which it is viewed, is called *parallax;* and it is always expressed by the angle at the object, which is subtended by the line joining the two points of view. (Hersch. Ast., Art. 70.) Thus in Fig. 17, the object S would be seen from A in the direction A S; and from C in the direction C S. The angle at S, subtended by A C, is the difference of these directions, or the parallax for the two points of view, C and A.

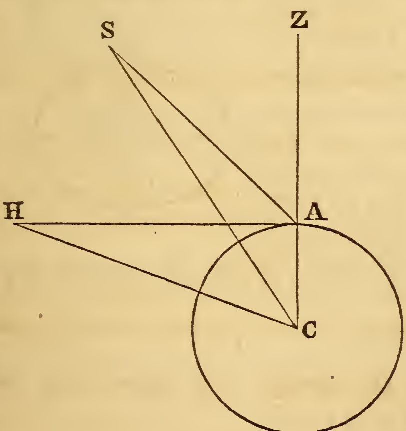

Fig. 17.

54. In astronomical observations, the observer is on the surface of the earth; the conventional point to which it is most convenient to reduce them, wherever they may be made, is the earth's centre. (Hersch. Ast., Art. 80.) In those problems of practical astronomy which are used by the navigator, we have only to con-

sider this *geocentric parallax*, which is the difference of the direction of a body seen from the surface and from the centre of the earth. It may also be defined to be the angle at the body subtended by that radius of the earth, which passes through the place of the observer. Thus, in Fig. 17, if

C is the centre of the earth, and
A the place of the observer,

the geocentric parallax of a body, S, will be the angle

$$S = Z\,A\,S - Z\,C\,S,$$

at the body subtended by the radius C A.

If the earth is regarded as a sphere, C A Z will be the vertical line through A, and will pass through the zenith Z. Then will the plane of C A S be a vertical plane;

Z A S, the *apparent* zenith distance of S as observed at A;
Z C S, its geocentric or *true* zenith distance; and
$Z\,A\,S > Z\,C\,S$.

Thus we see that this parallax takes place in a vertical plane, and increases the zenith distance, or decreases the altitude, of a heavenly body without affecting its azimuth.

55. This suffices for all nautical problems except the complete reduction of lunar distances. For these and the more refined observations at observatories, the spheroidal form of the earth must be considered. Then, as in Fig. 18, the radius C A does not coincide with the normal or vertical line C′ A Z, but meets the celestial sphere at a point Z′, in the celestial meridian, nearer the equator than the zenith, Z.

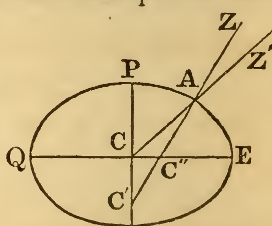

Fig 18.

We may remark here that

A C″ E, the angle which the vertical line makes with the equator, is the latitude of A; and

A C E, the angle which the radius makes with the equator, is its *geocentric* latitude.

56. PROBLEM 18. *To find the parallax of a heavenly body for a given altitude.*

Solution. In Fig. 17, let

$p =$ S, the parallax in altitude;
$z =$ Z A S, the apparent zenith distance of S, corrected for refraction;
$R =$ C A, the radius of the earth;
$d =$ C S, the distance of the body, S, from the centre of the earth.

Then from the triangle C A S, we have

$$\sin \text{C S A} = \frac{\text{C A}}{\text{C S}} \sin \text{C A S},$$

or

$$\sin p = \frac{R \sin z}{d}, \tag{58}$$

If the object is in the horizon as at H, the angle A H C is called its horizontal parallax; and denoting it by π, we have from (58), or from the right triangle C A H,

$$\sin \pi = \frac{R}{d}, \tag{59}$$

which substituted in (58) gives

$$\sin p = \sin \pi \sin z. \tag{60}$$

If $h = 90° - z$, the apparent altitude of the object, we have

$$\sin p = \sin \pi \cos h; \tag{61}$$

or, nearly, since p and π are small angles,

$$p = \pi \cos h. \tag{62}$$

57. The horizontal parallax, π, is given in the Nautical Almanac for the sun, moon, and planets. From Fig. 17 it is obviously the semidiameter of the earth, as viewed from the body. As the equatorial semidiameter is larger than any other, so also will be the *equatorial horizontal parallax.*

This is what is given in the Almanac for the moon. Strictly it requires reduction for the latitude of the observer, and such reduction is made at observatories, and in the higher order of astronomical observations.

58. Tables X. A and XIV. (Bowd.) are computed by formula (62).

Table XXIX. contains the correction of the moon's altitude for parallax and refraction corresponding to a mean value of the horizontal parallax, 57′ 30″. It should be used, however, only for very rough observations or a coarse approximation.

Tab. XIX. contains the difference of 59′ 42″ and the combined correction of the moon's altitude for parallax and refraction. The numbers taken from this table subtracted from 59′ 42″, give the correction of an apparent altitude for parallax and refraction. To this may be applied the reduction of the refraction to the actual condition of the atmosphere (Art. 42). If, instead of the equatorial hor. parallax, we enter the table with the augmented parallax of Chauvenet's Lunar Method (Tab. III.), we shall obtain the reduction, not to the centre of the earth, C (Fig. 18), but to the point, C′, where the normal line through A intersects the axis of the earth.

Table XIX. of Bowditch was arranged especially for one of the Lunar methods in that work, so that the reductions of the distance should all be additive.

APPARENT SEMIDIAMETERS.

59. The apparent diameter of a body is the angle which its disk subtends at the place of the observer.

PROBLEM 19. *To find the apparent semidiameter of a heavenly body.*

Solution. In Fig. 19, let M be the body;

$d =$ C M, its distance from the centre of the earth;

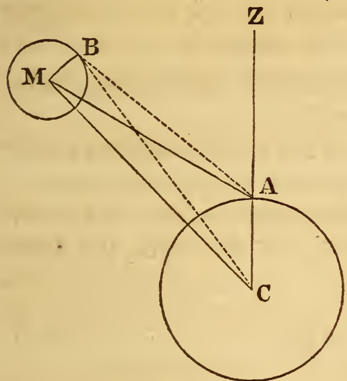

Fig. 19.

$d' = AM$, its distance from A;
$r = MB$, its linear radius or semidiameter;
$s = MCB$, its apparent semidiameter, as viewed from C;
$s' = MAB'$, its apparent semidiameter, as viewed from A (B and B′ are too near each other to be distinguished in the diagram);
$R = CA$, the earth's radius.

1st. For finding s, the right triangle C B M, gives

$$\sin s = \frac{r}{d}. \tag{63}$$

Were the body M in the horizon of A, or Z A M = 90°, its distance from A and C would be sensibly the same, so that the angle s is called the *horizontal* semidiameter.

In (59) we have for the horizontal parallax,

$$\sin \pi = \frac{R}{d}, \qquad \text{or } d = \frac{R}{\sin \pi},$$

which, substituted in (63), gives

$$\sin s = \frac{r}{R} \sin \pi, \tag{64}$$

or nearly, since s and π are small,

$$s = \frac{r}{R} \pi. \tag{65}$$

$\frac{r}{R}$ is constant for any particular body, as it is simply the ratio of its linear diameter to that of the earth. (Hersch. Ast., p. 544.)

For the moon (Hersch. Ast., p. 214),

$$\frac{r}{R} = 0.2729,$$

$$\left.\begin{aligned} s &= 0.2729\ \pi, \\ \text{and} \qquad \log s &= 9.43600 + \log \pi. \end{aligned}\right\} \tag{66}$$

By this formula the moon's horizontal semidiameter may be found from her horizontal parallax.

The Nautical Almanac contains the semidiameters as well as the horizontal parallaxes of the sun, moon, and planets.

2d. For finding s', the apparent semidiameter as viewed by an observer at A on the surface of the earth, the right triangle A B′ M gives

$$\sin s' = \frac{r}{d'}. \tag{67}$$

In the triangle C M A,

$$\frac{\sin \mathrm{M\,A\,C}}{\sin \mathrm{M\,C\,A}} = \frac{\mathrm{C\,M}}{\mathrm{A\,M}},$$

or, putting $h = 90° - \mathrm{Z\ A\ M}$, the apparent,
and $h' = 90° - \mathrm{Z\ C\ M}$, the true altitude of M,

$$\frac{\cos h}{\cos h'} = \frac{d}{d'}, \tag{68}$$

whence,

$$d' = d\frac{\cos h'}{\cos h},$$

which, substituted in (67), and by (63), gives

$$\sin s' = \frac{r \cos h}{d \cos h'} = \sin s \frac{\cos h}{\cos h'},$$

or approximately, $$s' = s\frac{\cos h}{\cos h'}, \tag{69}$$

by which s' may be found when s and h are known.

Since $h < h'$, $\cos h > \cos h'$, and consequently $s' > s$; that is, the semidiameter increases with the altitude of the body. The excess

$\Delta s = s' - s$, is called the augmentation.

The moon is the only body for which this augmentation is sensible.

60. Problem 20. *To find the augmentation of the moon's horizontal semidiameter.*

Solution. From (69) we find

$$\varDelta s = s' - s = s\frac{\cos h - \cos h'}{\cos h'},$$

which, by Pl. Trig. (108), becomes,

$$\varDelta s = s\frac{2 \sin \frac{1}{2}(h' + h) \sin \frac{1}{2}(h' - h)}{\cos h'}.$$

$h' - h = p$, the parallax; since it is small, we may put

$$2 \sin \tfrac{1}{2}(h' - h) = 2 \sin \tfrac{1}{2} p = p \sin 1'' = \pi \cos h \sin 1'';$$

and, in computing so small a quantity as $\varDelta s$, we may take h for $\frac{1}{2}(h' + h)$, and $\cos h$ for $\cos h'$; and then

$$\varDelta s = s\,\pi \sin 1'' \sin h,$$

or, since (65)

$$s = \frac{r}{R}\pi,$$

$$\varDelta s = \frac{r}{R}\pi^2 \sin 1'' \sin h.$$

For the moon

$$\frac{r}{R} = 0.2729; \text{ then}$$

$$\varDelta s = .000001323\,\pi^2 \sin h. \tag{70}$$

If we take $\pi = 57' \, 20''$, which is nearly its mean value, we have

$$\varDelta s = 15''.65 \sin h, \tag{71}$$

which agrees nearly with the formula for Tab. XV. (Bowd.) The augmentation may differ $2''$ from this mean value.

Tab. II. of Chauvenet's Lunar Method contains this augmentation for different values of s, as well as of h, computed by a more precise formula.

CHAPTER III.

TIME.

61. *Transit.* The instant when any point of the celestial sphere is on a given meridian is designated as the transit of the point over that meridian.

62. *Hour-angle.* The hour-angle of any point of the sphere is the angle at the pole, which the circle of declination passing through the point makes with the meridian. It is properly reckoned from the upper branch of the meridian, and positively toward the west. It is usually expressed in hours, minutes, and seconds of time. The intercepted arc of the equator is the measure of this angle.

63. *Sidereal Time.* The intervals between the successive transits of any fixed point of the sphere (as, for instance, of a star which has no proper motion) over the same meridian would be perfectly equal, were it not for the *variable* effect of nutation. (Hersch. Ast., Art. 327.) This correction, arising from a change in the position of the earth's axis, is most perceptible in its effect upon the transit of stars near the vanishing point of that axis, i. e. near the poles of the heavens. Hence, for the exact measurement of time, we use the transits of some point of the equator, as the *vernal equinox.* This point is often called the *first point of Aries.* Its usual symbol is ♈.

64. The interval between two successive transits of the vernal equinox is a *sidereal day;* and such a day is regarded as commencing at the instant of the transit of that point.

The sidereal time is then $0^h\ 0^m\ 0^s$. This instant is sometimes called *sidereal noon.*

The effect of nutation and precession in changing the time of the transit of the vernal equinox is so nearly the same at two successive transits, that the sidereal days thus defined are sensibly equal. It is unnecessary, then, except in refined discussions, to discriminate between *mean* and *apparent* sidereal time.

65. The *sidereal time* at any instant is the hour-angle of the vernal equinox at that instant, and is reckoned on the equator from the meridian westward around the entire circle, that is, from 0 to 24^h. It is equal to the right ascension of the meridian at the same instant.

66. *Solar Time.* The interval between two successive transits of the sun over a given meridian is a *solar day*, and the hour-angle of the sun at any instant is the *solar time* of that instant.

In consequence of the motion of the earth about the sun from west to east, the sun appears to have a like motion among the stars at such a rate that it increases its right ascension daily nearly 1°, or 4^m of time. With reference to the fixed stars, it therefore arrives at the meridian each day about 4^m later than on the previous day; consequently, solar days are about 4^m longer than sidereal days.

67. *Apparent and Mean Solar Time.* If the sun changed its right ascension uniformly each day, solar days would be exactly equal. But the sun's motion in right ascension is not uniform, varying from $3^m\ 35^s$ to $4^m\ 26^s$ in a solar day. There are two reasons for this,—

1st. The sun does not move in the equator, but in the ecliptic.

2d. Its motion in the ecliptic is not uniform, being most rapid at the time of the earth's perihelion, about January 1, and slowest at the time of the aphelion, about July 2.

To obtain a uniform measure of time depending on the sun's motion, the following method is adopted. A fictitious sun, called a *mean sun*, is supposed to move uniformly in the *ecliptic* at such a rate as to return to the perigee and apogee at the same time with the true sun. A *second mean sun* is also supposed to move uniformly in the *equator* at the same rate that the first moves in the ecliptic, and to return to each equinox at the same time with the first mean sun.

The time which is measured by the motion of this second mean sun is uniform in its increase, and is called *mean* time.

That which is denoted by the true sun is called *true* or *apparent* time.

The difference between mean and apparent time is called the *equation of time.* It is also the difference of the right ascensions of the true and mean suns.

The instant of transit of the true sun over a given meridian is called *apparent noon.* The instant of transit of the second mean sun is called *mean noon.* The mean time is then 0^h 0^m 0^s.

Mean noon occurs, then, sometimes before and sometimes after apparent noon, the greatest difference being about 16^m, early in November.

68. *Astronomical Time.* The solar day (apparent or mean) is regarded by astronomers as commencing at noon (apparent or mean), and is divided into 24 hours, numbered successively from 0 to 24.

Astronomical time (apparent or mean) is, then, the hour-angle of the sun (true or mean) reckoned on the equator *westward* throughout the entire circle from 0^h to 24^h.

69. *Civil Time.* For the common purposes of life, it is more convenient to begin the day at midnight, that is, when the sun is on the meridian below the horizon, or at the sun's lower transit. The civil day begins 12^h before the astronomical day of the same date; and is divided into two

periods of 12^h each, namely, from midnight to noon, marked A.M. (ante-meridian), and from noon to midnight, marked P.M. (post-meridian). Both apparent and mean time are used.

The affixes A.M. and P.M. distinguish civil time from astronomical time. During the P.M. period, this is the only distinction,—the day, hours, &c. being the same in both.

70. *Sea-Time.* Formerly, in sea-usage, the day was supposed to commence at noon, 12^h before the civil day, and 24^h before the astronomical day of the same date; and was divided into two periods, the same as the civil day. Sea-time is now rarely used.

71. *To convert civil into astronomical time*, it is only necessary to drop the A.M. or P.M., and when the civil time is A.M., deduct 1^d from the day and increase the hours by 12^h.

To convert astronomical into civil time, if the hours are less than 12^h, simply affix P.M.; if the hours are 12^h or more than 12^h, deduct 12^h, add 1^d, and affix A.M.

Examples.

Ast. Time.		d	h	m	s		Civil Time.		d	h	m	s	
1860	May	10	14	15	10	=	1860	May	11	2	15	10	A.M.
1862	Sept.	8	9	19	20	=	1862	Sept.	8	9	19	20	P.M.
1863	Jan.	3	23	22	16	=	1863	Jan.	4	11	22	16	A.M.
1863	Jan.	4	0	3	30	=	1863	Jan.	4	0	3	30	P.M.

72. The hour-angle of the sun (true or mean), at any meridian, is called the *local* (apparent or mean) solar time. The hour-angle of the sun (true or mean) at Greenwich at the same instant is the corresponding *Greenwich* time.

So also the hour-angle of ♈ at any meridian, and its hour-angle at Greenwich at the same instant, are corresponding *local* and *Greenwich* sidereal times.

73. *The difference of the local times of any two meridians is equal to the difference of longitude of those meridians.*

Demonstration. In Fig. 20, let

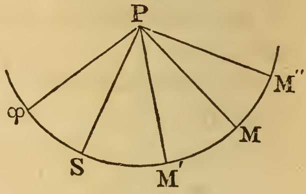

Fig. 20.

P M, P M′ be the celestial meridians of two places;

P S, the declination circle through the sun (true or mean);

M P S, the hour-angle of the sun at all places whose meridian is P M, will be the local time (apparent or mean) at those places; so also

M′ P S will be the corresponding local time at all places whose meridian is P M′; and

M P M′ = M P S − M′ P S will be the difference of longitude of the two meridians.

If P ♈ is the equinoctial colure,

M P ♈ and M′ P ♈ will be the corresponding sidereal times at the two meridians; still, however,

M P M′ = M P ♈ − M′ P ♈.

The proposition is true, then, whether the times compared are *apparent*, *mean*, or *sidereal*.

The difference of longitude is here expressed in time. It is readily reduced to arc by observing that

$$\left.\begin{array}{rl} 24^h &= 360^\circ \\ 1^h &= 15^\circ \\ 1^m &= 15' \\ 1^s &= 15'' \end{array}\right\} \text{ or } \left\{\begin{array}{l} 1^\circ = 4^m \\ 1' = 4^s \\ 1'' = \frac{1}{15}^s \end{array}\right.$$

In comparing corresponding times of different meridians, the most easterly meridian is that at which the time is *greatest*.

74. If (Fig. 20) P M is the meridian of Greenwich,

M P S is the Greenwich solar time, and

M P M′ the longitude of the meridian P M′.

$$\text{M P M}' = \text{M P S} - \text{M}' \text{P S};$$

so also $$\text{M P M}' = \text{M P} \,\text{♈} - \text{M}' \text{P} \,\text{♈};$$

or, *the longitude of any meridian is equal to the difference between the local time of that meridian and the corresponding Greenwich time.*

75. If we put

$T_0 = $ M P S, the Greenwich time,
$T = $ M′ P S, the corresponding local time,
$\lambda = $ M P M′, the longitude of the meridian, P M′,

we have and
$$\left.\begin{aligned} \lambda &= T_0 - T, \\ T_0 &= T + \lambda, \end{aligned}\right\} \qquad (72)$$

in which λ is + for west longitudes, and T_0 and T are supposed to be reckoned always *westward* from their respective meridians from 0^h to 24^h; that is, T_0 and T are the *astronomical* times, which should always be used in all astronomical computations.

76. Usually the *first* operation in most computations of nautical astronomy is to convert the local civil time into the corresponding astronomical time (Art. 71).

The Greenwich time should never be otherwise expressed than astronomically. On this account it would be convenient to have chronometers intended for nautical or astronomical purposes marked from 0^h to 24^h, instead of 0^h to 12^h as is now customary with sea-chronometers.

77. The *second* operation often required is to convert the local astronomical time into Greenwich time. For this we have (72), which numerically is

$$T_0 = T \pm \lambda \begin{cases} + \text{ when the longitude is } \textit{west}, \\ - \text{ when it is } \textit{east}, \end{cases}$$

and, in words, gives the following

Rule. Having expressed the local time astronomically, *add* the longitude, if *west; subtract* it, if *east:* the result is the corresponding Greenwich time.

TIME.

Examples.

1. In Long. 76° 32′ W., the local time being 1861, April $1^d\ 9^h\ 3^m\ 10^s$ A.M., what is the Greenwich time?

Local Ast. T. =	March	31^d	21^h	3^m	10^s
Longitude =		+	5	6	8
G. T. =	April	1	2	9	18

2. In Long. 30° E., the loc. time being March $20^d\ 6^h\ 3^m$ A.M., what is the G. T.?

Loc. Ast. T. =	March	19^d	18^h	3^m
Long. =		−	2	0
G. T. =	March	19	16	3

3. In Long. 105° 15′ E., the loc. time being Aug. $21^d\ 4^h\ 3^m$ P.M., what is the G. T.?

Loc. Ast. T. =	Aug.	21^d	4^h	3^m
Long. =		−	7	1
G. T. =	Aug.	20	21	2

4. Long. 175° 30′ W., Loc. T. Sept. $30^d\ 8^h\ 10^m$ A.M.
G. T. Sept. $30^d\ 7^h\ 52^m$.

5. Long. 165° 0′ E., Loc. T. Feb. $1^d\ 7^h\ 11^m$ P.M.
G. T. Jan. $31^d\ 20^h\ 11^m$.

6. Long. 72° 30′ W., Loc. T. April $10^d\ 7^h\ 10^m$ A.M.
G. T. $10^d\ 0^h\ 0^m$.

7. Long. 100° 30′ E., Loc. T. June $1^d\ 0^h\ 0^m$ A.M. (or midnight.)
G. T. May $31^d\ 5^h\ 18^m$.

8. Long. 75° W., Loc. T. June $3^d\ 0^h\ 0^m$ M. (noon.)
G. T. June $3^d\ 5^h\ 0^m$.

By reversing this process, that is by *subtracting* the longitude if *west*, or adding it if *east*, we may reduce the Greenwich time to the corresponding local time.

78. When observations are noted by a chronometer regulated to Greenwich time, an *approximate* knowledge of the

longitude and local time is necessary in order to determine whether the chronometer time is A.M. or P.M., and thus fix the true Greenwich date. If the time is A.M., the hours must be increased by 12^h.

Examples.

1. In Long. 5^h W., about 3^h P.M., on Aug. 3^d, the Greenwich chronometer shows $8^h\ 11^m\ 7^s$, and is fast of G. T. $6^m\ 10^s$. What is the G. T.?

Approx. Loc. T.	Aug. $3^d\ 3^h$	G. Chro.		$8^h\ 11^m\ 7^s$
Long.	$+\ 5$	Correction		$-\ 6\ 10$
Approx. G. T.	Aug. $3^d\ 8^h$	G. T. Aug.	3^d	$8^h\ 4^m\ 57^s$

2. In Long. 10^h E., about 1^h A.M., on Dec. 7^d, the G. Chro. shows $3^h\ 14^m\ 13^s.5$, and is fast $25^m\ 18^s.7$, find the G. T.

Approx. Loc. T.	Dec. $6^d\ 13^h$	G. Chro.		$3^h\ 14^m\ 13^s.5$
Long.	$-\ 10$	Correction		$-\ 25\ 18^s.7$
Approx. G. T.	Dec. $6^d\ 3^h$	G. T. Dec.	6^d	$2^h\ 48^m\ 54^s.8$

3. In Long. $9^h\ 12^m$ W., about 2^h A.M., on Feb. 13^d, the G. Chro. shows $11^h\ 27^m\ 13^s.3$, and is fast $30^m\ 30^s.3$, find the G. T.

Approx. Loc. T.	Feb. $12^d\ 14^h\ 0^m$	G. Chro.		$11^h\ 27^m\ 13^s.3$
Long.	$+\ 9\ 12$	Correction		$-\ 30\ 30^s.3$
Approx. G. T.	Feb. $12^d\ 23^h\ 12^m$	G. T. Feb.	12^d	$22^h\ 56^m\ 43^s.0$

The operations on the approximate times may be performed mentally.

CHAPTER IV.

THE NAUTICAL ALMANAC.

79. The American Ephemeris and Nautical Almanac "is divided into two distinct parts. One part is designed for the special use of navigators, and is adapted to the meridian of Greenwich. The other is suited to the convenience of astronomers, on this continent particularly, and is adapted to the meridian of Washington."

80. The Nautical part of this Ephemeris and the British Nautical Almanac give at regular intervals of *Greenwich* time the *apparent* right ascensions and declinations of the sun, moon, planets, and principal fixed stars, the equation of time, the horizontal parallaxes and semidiameters of the sun, moon, and planets, and other quantities, some of which little concern the navigator, but are needed by astronomers.

81. Before we can find the value of any of these quantities for a given *local* time, we must first find the corresponding *Greenwich* time (Art. 77). When this time is exactly one of the instants for which the required quantity is put down in the Almanac, it is only necessary to transcribe the quantity as it is there given. When, as is mostly the case, the time falls between two Almanac dates, the required quantity is to be obtained by interpolation. And generally, except when great precision is desired, it is sufficient to use *first* differences only; that is, *regard the changes of the quantity as proportional to the small intervals of time*, which are employed.

Thus, for a day the change of the sun's right ascension may be regarded as uniform, so that for 1^h it is $\frac{1}{24}$ of the daily change; for 2^h, $\frac{2}{24}$; and in general for any part of a day it will be the same part of the daily change.

Generally, then, if

A_0 represent the quantity in the Almanac, for a date *preceding* the given Greenwich time;

Δ_1, its change in the time T;

t, the time *after* the Almanac date for which the value of the quantity is required, expressed in the same unit as T; and

A, the required value;

we have,

$$A = A_0 + \frac{t}{T}\,\Delta_1. \tag{73}$$

When A_0 is increasing, Δ_1 has the same sign as A_0; but when A_0 is decreasing, Δ_1 has the opposite sign.

82. If the given time is nearer the subsequent than the preceding Almanac date, it may be convenient to interpolate backward. If, then, A_1 represent the quantity in the Almanac for a *subsequent* Greenwich date, and t' the time *before* the Almanac date, we have

$$A = A_1 - \frac{t'}{T}\,\Delta_1. \tag{74}$$

83. The Almanac contains the *rate of change*, or *difference* of each of the principal quantities for some *unit* of time. Thus, in the Ephemeris of the sun and planets the "Diff. for 1^h," in part of that of the moon, the "Diff. for 1^m," are given. If t or t' is expressed in the same unit of time as that for which the "Diff.," Δ_1, is given, formulas (73) and (74) become

$$\left.\begin{aligned} A &= A_0 + t\,\Delta_1, \\ A &= A_1 - t'\,\Delta_1. \end{aligned}\right\} \tag{75}$$

Thus, for using *hourly* differences, we wish the hours,

minutes, &c., of the Greenwich time expressed in hours and parts of an hour; for using the differences for 1^m, we wish the minutes and seconds of Greenwich time expressed in minutes and parts of a minute. *Decimal* parts are usually most convenient; though some computers prefer *aliquot* parts.

84. The quantities in the Almanac, as commonly in other mathematical tables, are approximate numbers, that is, each is given only to the nearest unit of the lowest retained order; and no refinement of interpolation can give a result to a higher degree of precision. In interpolating, more than one lower order in any case is superfluous. Thus, the sun's declination is given to the nearest $0''.1$, and in no way can we by interpolation obtain a value which will be reliable within a narrower limit.

Moreover, the Greenwich times are uncertain to a greater or less extent; and if *first* differences only are used, the interpolated result can be regarded as true only within much wider limits than the approximation of the Ephemeris.

In interpolating, then, it is well to consider the degree of approximation which is wanted in any particular case; and if the nearest $1'$, or $10''$, or $1''$ suffices, contract the interpolation so as to retain at the most one lower order; or else, consider the degree of approximation attainable in any particular case, and contract the work so as to retain only the reliable figures. All lower orders are superfluous, and are deceptive, as giving the appearance of a higher degree of accuracy than has actually been obtained; as, for instance, using *tenths* and *hundredths* of seconds, when the data will give a result reliable within $2'$ or $3'$ only.

A convenient method of contracting the multiplication and division of decimals is given in a pamphlet on the subject.

85. Should it be desirable to interpolate more accurately

than can be done by first differences alone, the reduction for *second* differences may be introduced by a simple process.

Let Δ_2 be the change of Δ_1 in the time T', then instead of Δ_1, as found in the Almanac for the nearest Greenwich date, we may substitute

$$\Delta_1 + \frac{t}{2\,T'}\Delta_2; \tag{76}$$

that is, the value of Δ_1, interpolated for $\frac{1}{2}\,t$, or to the middle instant between the Almanac date and the given time. This is simply using the mean rate of change during the interval.

If Δ_1 is a "Diff. for 1^h" given for the Almanac for each day, $T' = 24^h$; if Δ_1 is a "Diff. for 1^m" given in the Almanac for each hour, $T' = 60^m$.

The interpolation of Δ_1 to the middle instant may often be performed mentally.

Example. If the sun's right ascension for 1865, Jan. 30, $8^h\ 9^m$ time be required, we find in the Almanac,

$$\begin{array}{lll} \text{for Jan. 30 } 0^h & \Delta_1 = 10''.246 & \\ & & \Delta_2 = -0''.035 \\ \qquad\quad 31\ 0^h & \Delta_1 = \ 10.211 & \end{array}$$

and by interpolation for Jan. 30 4^h, the middle instant between Jan. 30 0^h and Jan. 30 8^h,

$$\Delta_1 = 10''.246 - 0''.006 = 10''.240,$$

which is the mean hourly change in the interval from 0^h to 8^h.

86. Formula (76), however, applies only to the American Ephemeris, where the differences for 1^h or for 1^m, which are designated by Δ_1, are given for the same instants of Greenwich time as the functions, A, to which they belong.*

In the British Almanac they are given for the middle instant between two dates.† For instance, the "Diff. for 1^h"

* The "Prop. Logs. of Diff." of the Lunar Distances are given for the middle instant.

† In the Ephemeris of the Planets it is otherwise.

given in each as if for *noon* Jan. 1^d, is in the American Ephemeris the change per hour at Jan. $1^d\ 0^h$; in the British Almanac, the change per hour, Jan. $1^d\ 12^h$, or *midnight.*

For the British Almanac (76) becomes

$$\Delta_1 + \frac{t - T'}{2\,T'}\,\Delta_2 \qquad (77)$$

Δ_1 being taken from the same line or for the same date as A_0. This is the date preceding that of A_1.

87. PROBLEM 21. *To find from the Almanac a required quantity for a given mean time at a given place.*

Solution. The preceding considerations lead to the following rule:—

1. Express the given *mean* time astronomically, stating the day as well as the hours, &c., and reduce it to Greenwich mean time by *adding* the longitude if *west*, *subtracting* if *east*.

2. Take from the Almanac for the nearest *preceding* mean time date the required quantity and the corresponding "Diff. for 1^h," or "Diff. for 1^m," noting the name or sign of each; multiply the "Diff. for 1^h" by the hours and parts of an hour, or the "Diff. for 1^m" by the minutes and parts of a minute, of the remaining Greenwich time; and *add* the product algebraically.

Or, take out for the nearest *subsequent* date the required quantity and its difference;* multiply the "Diff." by the hours and parts of an hour, or the minutes and parts of a minute, of the interval from the given Greenwich date to the Almanac date; and *subtract* the product algebraically.

When greater precision is required, interpolate the *difference* to the middle instant between the given Greenwich date and the Almanac date, and use the result instead of the difference given in the Almanac.

---

* From the British Almanac, the difference given for the *preceding* date should be taken.

This rule is applicable to all those quantities, which are given at regular intervals of Greenwich *mean* time, except the moon's meridian passage and age and lunar distances.

For the "Sidereal Time at Greenwich Mean Noon," on p. II. of each month, the "Diff. for 1^h" is $9^s.856$; the part of Tab. II. of the American Ephemeris, *for converting a mean solar into a sidereal interval*, may be used for the interpolation.

The "Mean Time of Sidereal 0^h," on p. III., is given at intervals of 24^h of *sidereal* time. The "Diff. for 1^h" is $-9^s.830$; and the part of Tab. II. *for converting a sidereal into a mean solar interval* may be used.

88. The quantities given in the American Ephemeris for Washington mean time may be interpolated in the same way, by reducing the local time to Washington time instead of to Greenwich time.

89. The apparent places of the fixed stars are given in the British Almanac for the upper transit over the meridian of Greenwich; in the American, for the upper transit over the meridian of Washington. In the latter, the Washington mean time is given. The *sidereal* time at either place for the instant of transit is the right ascension of the star (Art. 65).

Generally, the position given for the nearest day suffices. But if greater precision is required, it is necessary to reduce the local mean time to the sidereal time of the prime meridian, and interpolate for it.

90. In the following examples, the required quantities are taken from the American Ephemeris, and interpolated to the nearest second by 1st differences (75), and to the highest precision attainable by 2d differences (76).

Examples.

For the local *mean* time, 1865, Jan 30^d 9^h 14^m 30^s A.M. in Long. 163° 14′ W., find the following quantities from the Nautical Almanac:—

The equation of time.

☉'s right ascension,	☽'s right ascension,
☉'s declination,	☽'s declination,
☉'s semidiameter,	☽'s horizontal parallax,
☉'s horizontal parallax;	☽'s semidiameter;

Jupiter's right ascension, declination, horizontal parallax, and semidiameter;

The R. ascension and declination of *a* Scorpii (*Antares*).

Ast. mean time, 1865, Jan.	29^d	21^h 14^m 30^s
Long.		+10 52 56
G. mean time, 1865,	Jan. 30	8 7 26
		8 7.433
		8.1239

1. *The Equation of Time.* (Page II.)

Jan. 30 0^h	13^m 38.8^s	$+0.390^s$ in 1^h		13^m 38.81^s	$+0.384^s$ (at 4^h)	
		3.1 in 8			3.072	in 8^h
	+ 3.2	.1	.124	+ 3.12	38	.1
					8	.02
	13 42			13 41.93	2	.004

subtractive from mean time.

2. *The* ☉*'s right ascension.* (Page II.)

Jan. 30 0^h	20^h 52^m 34.8^s	$+10.246^s$		20^h 52^m 34.81^s	$+10.240^s$ (at 4^h)	
		82.0	in 8^h		81.920	in 8^h
	+ 1 23.3	1.0	.1	+ 1 23.19	1.024	.1
		.3	.024		.205	.02
	20 53 58			20 53 58.00	41	.004

3. *The* ☉*'s declination.* (Page II.)

Jan. 30 0^h	−17° 33′ 56.7″	+41.44″		−17° 33′ 56.7″	+41.57″ (at 4^h)	
		331.5	in 8^h		332.56	in 8^h
	+ 5 36.6	4.1	.1	+ 5 37.7	4.16	.1
		.8	.02		.83	.02
	−17 28 20	.2	.004	−17 28 19.0	.17	.004

4. *The* ⊙*'s semidiameter* (p. I.) and *hor. parallax* (p. 250).

$$\begin{array}{llll}
\text{Jan. } 30\ 0^{h} & 16'\ 16''.22\ \ -\overline{0''.14 \text{ in } 1^{d}} & & \odot\text{'s mean H. P. } 8''.5776 \\
 & -\ .05 \text{ in } \overline{8^{h}} = \tfrac{1}{3}^{d} & & \\
 & \underline{16\ 16.17} & & \text{for Jan. 30} \quad \underline{8.71}
\end{array}$$

5. *The* ☽*'s right ascension.* (Page XII.)

$$\begin{array}{lllllll}
\text{Jan. } 30\ 8^{h} & 23^{h}\ 50^{m}\ 28^{s}.8 & +\overline{2^{s}.3127} & & 23^{h}\ 50^{m}\ 28^{s}.77 & +\overline{2^{h}.3127} \text{ (at } 3^{m}.7) & \\
 & +16.2 & \left\{\begin{array}{ll} 15.2 & \text{in } 7^{m} \\ .9 & .4 \\ .1 & .033 \end{array}\right. & & +16.19 & \left\{\begin{array}{ll} 15.189 & \text{in } 7^{m} \\ .925 & .4 \\ 69 & .03 \\ 7 & .003 \end{array}\right. & \\
 & \underline{23\ 50\ 45} & & & \underline{23\ 50\ 44.96} & &
\end{array}$$

6. *The* ☽*'s declination.* (Page XII.)

$$\begin{array}{llllll}
\text{Jan. } 30\ 8^{h} & +2^{\circ}\ 7'\ 48''.5 & +\overline{12''.178} & & +2^{\circ}\ 7'\ 48''.5 & +\overline{12''.177} \text{ (at } 3^{m}.7) \\
 & +1\ 30.5 & \left\{\begin{array}{ll} 85.2 & \text{in } 7^{m} \\ 4.9 & .04 \\ .4 & .033 \end{array}\right. & & +1\ 30.5 & \left\{\begin{array}{ll} 85.24 & \text{in } 7^{m} \\ 4.87 & .04 \\ .41 & .033 \end{array}\right. \\
 & \underline{+2\ 9\ 19} & & & \underline{+2\ 9\ 19.0} &
\end{array}$$

7. *The* ☽*'s horizontal parallax.* (Page IV.)

$$\begin{array}{llllll}
\text{Jan. } 30\ 0^{h} & 60'\ 25''.7 & -\overline{0''.84} & & 60'\ 25''.7 & -\overline{0''.93} \text{ (at } 4^{h}) \\
 & -\ 6.8 & \left\{\begin{array}{ll} 6.7 & \text{in } 8^{h} \\ .1 & .12 \end{array}\right. & & -7.6 & \left\{\begin{array}{ll} 7.44 & \text{in } 8^{h} \\ .11 & .12 \end{array}\right. \\
 & \underline{60\ 19} & & & \underline{60\ 18.1} &
\end{array}$$

8. *The* ☽*'s semidiameter.* (Page IV.)

$$\begin{array}{llll}
\text{Jan. } 30\ 0^{h} & 16'\ 29''.7\ -\overline{5''.5 \text{ in } 1^{d}} & & 16'\ 29''.7 \\
 & -\ 1.8 \text{ in } \overline{8^{h}} = \tfrac{1}{3}^{d} & & -2.1 = -7''.55 \times .2729 \\
 & \underline{16\ 28} & & \underline{16\ 27.6}
\end{array}$$

In Art. (59) we have for the moon, $s = .2729\,\pi$; whence

$$\varDelta s = .2729\ \varDelta\pi:$$

so that the reduction of the semidiameter may be readily found by multiplying that of the horizontal parallax by .2729, as in the above example. This coëfficient admits of a convenient set of aliquot parts; for $.2730 = .25 + .025 - .002$, so that $\varDelta s = (\frac{1}{4} + \frac{1}{40} - \frac{1}{500})\ \varDelta\pi$ nearly.

9. ♃'*s right ascension.* (Page 230.)

Jan. 30 0^h	$17^h\ 22^m\ 24^s.4$	$+1^s.974$		$17^h\ 22^m\ 24^s.40$	$+1^s.971$ (at 4^h)
		15.8 in 8^h			15.768 in 8^h
	+16.0	.2 .12		+16.01	.197 .1
					39 .02
	17 22 40			17 22 40.41	8 .004

10. ♃'*s declination.* (Page 230.)

Jan. 30 0^h	$-22°\ 39'\ 32''.4$	$-1''.85$	$-22°\ 39'\ 32''.4$	$-1''.84$
		14.8 in 8^h		14.72 in 8^h
	−15.0	.2 .1	−14.9	.18 .1
				4 .02
	−22 39 47		−22 39 47.3	

11. ♃'s *semidiameter and horizontal parallax.* (Page 384.)

Jan. 30, Vert. sem. diam., $16''.29$, Hor. Par., $1''.46$.

This is the vertical semidiameter when the planet is on the meridian, or the semidiameter in the direction of the declination circle of the planet. The polar, or minor, semidiameter of the elliptic disk is given on page 230.

12. *The right ascension and declination of a Scorpii.* (Antares.)

The Washington mean time is Jan. 30 $2^h\ 59^m$, or Jan. 30.12. On page 258, which serves as an index, the *mean* R. A. is $16^h\ 21^m$. The *apparent* R. A. and Dec. are for Jan. 30.8 m. t. Washington,

	R. A. $16^h\ 21^m\ 8^s.67$	$+\ 0^s.33$	Dec. $-26°\ 7'\ 37''.1$	$-0''8.$
Change in $-0^d.7$	$-\ .02$		$+\ .1$	
	16 21 8.65		$-26\ \ 7\ \ 37.0$	

91. Problem 22. *To find from the Almanac the sun's right ascension and declination, and the equation of time for a given apparent time at a given place.*

Solution. This differs from the preceding problem simply in using the apparent instead of the mean time, and in taking the quantities from page I. for the month, where they are given for apparent noon, instead of from page II., where they are given for mean noon.

Examples.

Find the ⊙'s R. A. and Dec. and the equation of time for 1865 Jan. 30 $9^h\ 0^m\ 48^s$ A.M. apparent time in Long. 163° 14′ W.

Ast. app. time 1865 Jan.	29	21^h	0^m	48^s
Long.	+	10	52	56
G. app. time	29	7	53	44
		7	53.73	
		7.896		

	h m s	s
⊙'s R. A.	20 52 37.1	+ 10.246
	+ 1 20.9	{ 71.7, 9.2 }
	20 53 58	

	° ′ ″	″
⊙'s dec.	— 17 33 47.2	+ 41.44
	+ 5 27.2	{ 290.1, 33.1, 3.7, .3 }
	— 17 28 20	

	m s	s
Equation of time	+ 13 38.9	+ 0.390
	+ 3.1	
	+ 13 42.0	

92. Problem 23. *To find the right ascension and declination of the sun, and the equation of time at apparent noon of a given place, or when the sun is on the meridian.*

Solution. The local apparent time is $0^h\ 0^m\ 0^s$. The Greenwich apparent time is then equal to the longitude if *west*, that is, it is *after* the noon of the same date by a number of hours, &c., equal to the longitude. If the longitude is *east*, the Greenwich apparent time is *before* the noon of the same date by a number of hours, &c., equal to the longitude.

Hence, take these quantities from the Almanac for Greenwich apparent noon (page I.) of the same day as the local (civil) day, and apply a correction equal to the hourly difference multiplied by the hours and parts of an hour of the longitude; observing to add or subtract the correction, according as the numbers in the Almanac may require, for a time *after* noon, if the longitude is *west;* for a time *before* noon, if the longitude is *east.*

The hourly differences from the British Almanac should be taken as given for the preceding instead of the same day in *east* longitude.

EXAMPLES.

1. Find the sun's right ascension and declination, and the equation of time for apparent noon, 1865, Jan. 30, in Long. 163° 14′ W.

Long. $+10^h\ 52^m\ 56^s$

☉'s R. A. $20^h\ 32^m\ 37^s.14 + 10^s.238$

102.38 in 10^h
5.129 30^m
+1 51.36 3.413 20
.427 2 30^s
20 34 28.50 7 25

☉'s dec. $-17°\ 33'\ 47''.2\ \ +41''.62$

416.2
+7 32.6 20.81
13.87
−17 26 14.6 1.73
3

Eq. of T. $+13^m\ 38^s.90 + 0^s.382$

3.82 in 10^h
+4.15 .191 30^m
.127 20
+13 43.05 .016 $2\frac{1}{2}$

2. For apparent noon, 1865, March 21, in Long. 163° 14′ E.

Long. $-10^h\ 52^m\ 56^s$
−10 52.93
−10.882

☉'s R. A. $-0^h\ 3^m\ 20^s.43\ \ +9^s.099$

90.99
− 1 39.02 7.279
.728
0 1 41.41 18

☉'s dec. $+0°\ 21'\ 44''.9 + 59''.21$

592.1
− 10 44.3 47.37
4.74
+0 11 0.6 .12

Eq. of T. $+7^m\ 15^s.59\ \ -0^s.757$

7.57
+8.24 .606
61
+7 23.83 2

3. For apparent noon, 1865, March 20, in Long. 150° 35′ W.

Long. $+10^h\ 2^m\ 20^s$

☉'s R. A. $23^h\ 59^m\ 42^s.07 +\ 9^s.101$

91.01 in 10^h
+ 1 31.36 .303 2^m
51 20^s
0 1 13.43

⊙'s dec. $-0^{\circ}\ 1'\ 56''.2 + 59''.23$	Eq. of T. $+7^{m}\ 33^{s}.72 - 0^{s}.755$
592.3 in 10^{h}	75.5
$+5\ 54.6$ $\quad 1.97 \quad 2^{m}$	$-\ 7.58$ $\quad 25$
$.33 \quad 20^{s}$	4
$+0\ \ 3\ 58.4$	$+7\ 26.14$

In the 1st and 2d examples, the Diffs. for 1^{h} have been interpolated for $5^{h}.5$ or half the longitude, forward in the first, back in the second: in the third they have been interpolated forward for 5^{h}.

93. Problem 24. *To find the right ascension of the mean sun for a given time and place.*

Solution. At the instant of mean noon, or when the mean sun is on the meridian, at any place, the right ascension of the mean sun is equal to the sidereal time. The quantity on page II. of each month, in the Almanac, called "sidereal time," is also the *right ascension of the mean sun* at Greenwich mean noon, and may be interpolated for a given local time in the same way as the right ascension of the true sun. (Prob. 21.) The constant "Diff. for 1^{h}" is $9^{s}.856$. A table for converting mean time into sidereal time intervals (Tab. II.) facilitates the interpolation.

We have also the right ascension of the mean sun equal to that of the true sun + the equation of time, using for the equation of time the sign of its application to *mean* time.

94. Problem 25. *To find the mean time of the moon's transit over a given meridian on a given day.*

Solution. The Almanac contains the mean time of each transit of the moon over the meridian of Greenwich (page IV.). This mean time is the hour-angle of the mean sun (Art. 72) when the moon is on the meridian; and is therefore the difference of right ascension of the moon and the mean sun. As this difference is constantly increasing, in consequence of the moon's more rapid increase of right

ascension, the mean time of each transit is later than that of the one preceding by a number of minutes, varying, according to the rate of the moon's motion, from 40^m to 66^m.

If, then, T_1 and T_2 denote the mean times of two successive transits of the moon over the Greenwich meridian, $T_2 - T_1$ is the *retardation* of the moon in passing over 24^h of longitude; so that for any longitude λ (expressed in hours) the retardation is nearly

$$\frac{\lambda}{24}(T_2 - T_1). \tag{78}$$

The mean time of a transit is, then, reduced from the Greenwich to any other meridian by interpolating for the longitude; *forward*, if the longitude is *west; backward*, if the longitude is *east*, since east longitudes are regarded as negative.

The American Ephemeris gives also the hourly differences, which facilitate the interpolation. For greater exactness, these differences may be interpolated for *half* the longitude. The practical rule will be :—

Take from the Almanac the mean time of meridian passage for the given *astronomical** day, and *add* to it the product of the " Diff. for 1^h" by the longitude in hours, if the longitude is *west; subtract* that product if the longitude is *east.*

From the British Almanac the *daily* retardation may be found by taking the difference for two successive transits; and the reduction by multiplying it by the longitude in parts of a day; or it may be taken from Tab. XXVIII. (Bowd.) The mean time of meridian passage for the given day, and that for the day *following* in *west* longitude, or for the day *preceding* in *east* longitude, are those which are commonly

* It is important to notice whether the mean time of transit is more or less than 12^h. In the former case, the astronomical day is 1^d less than the civil day.

used. (Bowd., p. 170.) But it is more exact to use half the difference of the times of meridian passage for the day preceding and the day following the given day: $\frac{1}{24}$ of this is the "Diff. for 1^h" of the American Ephemeris.

The times of transit are given only to tenths of a minute, which suffices the purposes of the navigator. They may be found more exactly for any meridian by the method hereafter given in Prob. 33.

95. PROBLEM 26. *To find on a given day the mean time of transit of a planet over a given meridian.*

Solution. The mean time of each meridian passage at Greenwich is given, in the Almanac, for each planet. It may be reduced to any meridian in the same way as for the moon; except that, in the case of an *acceleration*, the sign of the reduction is reversed.

EXAMPLES.

1. In Long. 100° 15′ W., find the times of meridian passage of the moon and Jupiter for 1865, June 6 (civil day).

Long. + $6^h\ 41^m\ 0^s = 6^h.683$.

	☽			♃		
	h m	s		h m	m	
M. T. of mer. pass. June 6	9 59.6	+1.99	June 5	12 41.8	−4.45 in 1^d	
		11.94			1.11 in 6^h	
	+13.3	1.19		− 1.2	.11	.6
		.16			1	.08
June 6	10 12.9	1	June 5	12 40.6		
			or June 6	0 40.6 A.M.		

2. In Long. 100° 15′ E. for 1865, June 6 (civil day), find the times of meridian passage of the moon and Jupiter.

Long. − $6^h\ 41^m\ 0^s = -\ 6^h.683$.

	☽			♃		
	h s	s		h m	m	
M. T. of mer. pass. June 6	9 59.6	+1.97	June 5	12 41.8	−4.45 in 1^d	
		11.82			1.11 in 6^h	
	−13.2	1.18		+ 1.2	.11	.6
		.16			1	.08
June 6	9 46.4	1	June 5	12 43.0		
			or June 6	0 43.0 A.M.		

In the case of the moon the hourly differences have been interpolated for half the longitude.

96. PROBLEM 27. *To find the right ascension or declination of the moon, or a planet, at the time of its transit over a given meridian on a given day.*

Solution. Find the local mean time of transit, as in Problem 25; deduce the corresponding Greenwich time by applying the longitude; and for this Greenwich time take out the right ascension or declination, as in Problem 21.

If the time of transit has been noted by a clock or chronometer, regulated to either local or Greenwich time, it should be used in preference to the time of transit computed from the Almanac.

97. PROBLEM 28. *To find the Greenwich mean time of a given lunar distance.*

Solution. The angular distances of the moon from the sun, the principal planets, and several selected stars, are given in the Almanac for each 3^h of Greenwich mean time.

If d represent the given distance;

d_0, the nearest distance of the same body in the Almanac preceding in time the given distance;

Δ_1, the change of distance in 3^h;

t, the required time (in hours) from the date of d_0;

by (75) we have approximately, using 1st differences only,

$$d = d_0 + \frac{t}{3^h}\Delta_1,$$

whence, for the inverse interpolation,

$$t = \frac{3^h}{\Delta_1}(d - d_0), \tag{79}$$

or, with t in seconds of time, which is better for computation,

$$t = \frac{10800^s}{\Delta_1}(d - d_0), \tag{80}$$

in which it is most convenient to express Δ_1 and $(d - d_0)$ in seconds.

Then by logarithms :

$$\log t = \log (d - d_0) + \log \frac{10800}{\Delta_1}, \qquad (81)$$

$\frac{\Delta_1}{10800}$ is the change of distance in 1^s; hence $\log \frac{10800}{\Delta_1}$ is the ar. complement of the "log diff. for 1^s."

It is given in the Almanac for the *middle* instant between the tabulated distances under the head "P. L.* of Diff."; the index, which is 0, and the separatrix being omitted.

In the same way, if

d_1 represent the distance in the Almanac following the given distance ; and

t', the interval *before* the date of d_1,

we shall have by (76

$$d = d_1 - \frac{t'}{3^h} \Delta_1,$$

and

$$t' = \frac{3^h}{\Delta_1} (d_1 - d),$$

or with t' in seconds, and by logarithms,

$$\log t' = \log (d_1 - d) + \log \frac{10800}{\Delta_1}. \qquad (82)$$

The computation is simplified by using a table of "logarithms of small arcs in space or time."† It differs from the common table of logarithms only in having the argument in *sexagesimal* instead of *natural* numbers. With such a table it is unnecessary to reduce differences of distance to seconds, or to first find the intervals of time in seconds.

From (81) and (82) we have the following rule : Find in the Almanac the two distances between which the given distance falls ; take out the nearest of these, the hours of Greenwich time over it and the "P. L. of Diff." between

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* Proportional Logarithm.

† Tab. I. of the American Ephemeris before 1865 : Tab. IX. of Chauvenet's Lunar Method.

them. Find the difference between the distance taken from the Almanac and the given distance; and to the log. of this difference add the "P. L. of Diff." from the Almanac; the sum is the log. of an interval of time to be *added* to the hours of Greenwich time taken from the Almanac, when the *earlier* Almanac distance is used; to be *subtracted* from the hours of Greenwich time when the *later* Almanac distance is used. (Chauvenet's Lunar Method, p. 8.)

98. The result, however, may not be sufficiently approximate, owing to the neglect of 2d differences. To correct it for 2d differences, Tab. X. of Chauvenet's Method, Tab. II. of the Almanac, or the table on p. 245 of Bowditch, may be used. For either, take the difference between the two Prop. Logs., which precede and follow the one taken from the Almanac. With half this difference, and the interval of time just found, enter the table and take out the seconds, which are to be *added* to the approximate Greenwich time when the Prop. Logs. are *decreasing*, but *subtracted* when they are *increasing*.

2d differences may also be introduced by first finding, or estimating, the Greenwich mean time to the nearest 10^m, and interpolating the Prop. Log. in the Almanac to the middle instant between that time and the Almanac hour used, as in Art. 88 for direct interpolation.

99. Maskelyne, the author of the present arrangement of Lunar distances, to facilitate their interpolation, devised what he chose to call *proportional logarithms*.

If n represent any number of seconds, either of space or time, the *proportional logarithm* of n is the log of $\frac{10800}{n}$.

Tab. XXII. (Bowd.) contains these proportional logarithms for each second of n from 0 to 3°, or to 3^h, the argument being in $^\circ\ '\ ''$ or in $^{h\ m\ s}$. But such a table is less useful for other purposes than Tab. I. of the American Ephemeris, previously referred to.

Dividing both members of (80) by 10800, and inverting, we have

$$\frac{10800}{t} = \frac{\Delta_1}{10800} \times \frac{10800}{d-d_0},$$

and,

$$\text{P. log } t = \text{P. log } (d-d_0) - \text{P. log } \Delta_1, \qquad (83)$$

which accords with the rule on page 231. (Bowd.)

100. Examples.

1865, Oct. 31, the distance of Fomalhaut from the moon's centre is 42° 3′ 35″, what is the Greenwich mean time?

	$d =$	42°	3′	35″	
Oct. 31 15^h	$d_0 =$	41	17	58	P. log 0.3142 ½ diff. −91
	$d - d_0 =$		45	37	log 3.4373
	$t =$	$+1^h$	34^m	3^s	log 3.7515
Red. for 2d diff.			+	28	
G. m. time	Oct. 31	16	34	31	

or, by back interpolation,

	$d =$	42°	3′	35″	
Oct. 31 18^h	$d_1 =$	42	45	17	P. log 0.3142 ½ diff. −91
	$d_1 - d =$		41	42	log 3.3983
	$t' =$	-1^h	25^m	58^s	log 3.7125
Red. for 2d diff.			+	28	
G. m. time	Oct. 31	16	34	30	

The P. L. interpolated to $15^h\ 47^m$ is 0.3163, and to $17^h\ 17^m$ is 0.3118. Had these been used instead of 0.3142, the resulting values of t and t' would have included the reduction for 2d difference.

CHAPTER V.

CONVERSION OF THE SEVERAL KINDS OF TIME.—RELATION OF TIME AND HOUR-ANGLES.

CONVERSION OF TIME

101. PROBLEM 29. *To convert apparent into mean time, or mean into apparent time.*

Solution. For the same instant, let

T_m represent the local mean time;
T_a, the local apparent time; and
E, the equation of time with the sign of its application to *apparent* time.

Then, since the equation of time is the difference of mean and apparent times (Art. 67),

$$\left.\begin{aligned} T_m &= T_a + E, \\ T_a &= T_m - E \end{aligned}\right\} \tag{84}$$

The reduction, then, is made by finding from the Almanac the equation of time for a given apparent time, from page I. of the month (Prob. 22), or for a given mean time from page II. (Prob. 21), and applying it to the given time according to the precept at the head of the column where it is found.

102. The equation of time on page I. is sometimes called the *mean time of apparent noon;* and on page II. the *apparent time of mean noon.* Regarding it, as in (84), as the reduction of apparent to mean time, it indicates, when additive

and increasing, or subtractive and decreasing, that mean time is *gaining* on apparent time.

103. PROBLEM 30. *To convert a mean into a sidereal time interval, or a sidereal into a mean time interval.*

Solution. The sidereal year is 365.25636 mean solar days, or 366.25636 sidereal days; so that the same interval of time which is measured by $365^d.25636$ reckoned in *mean* time, is measured by $366^d.25636$ if reckoned in *sidereal* time (Hersch., Ast. 305). Since both are uniform measures of time, if we represent any interval by

t, if expressed in *mean* time,

s, if expressed in *sidereal* time, then

$$\frac{s}{t} = \frac{366.25636}{365.25636} = 1.0027379\,;$$

whence

$$s = 1.0027379\ t = t + .0027379\ t, \tag{85}$$

$$t = 0.9972696\ s = s - .0027304\ s, \tag{86}$$

by which the reduction from one to the other may be made.

The computation is facilitated by Tab. II. of the American Ephemeris, the first part of which, for converting *sidereal into mean solar time*, contains for each minute of s the value of .0027304 s; the second part, for converting *mean solar into sidereal time*, contains for each minute of t the value of .0027379 t.

Tables LI. and LII. (Bowd.) contain the same quantities to tenths of seconds only.

104. If in (86) $t = 24^h$; $s = 24^h\ 3^m\ 56^s.5553$; or in a *mean* solar day sidereal time *gains* on mean time $3^m\ 56^s.5553$. In 1^h of mean time the gain is $9^s.8565$.

If in (87) $s = 24^h$; $t = 24^h - 3^m\ 55^s.9094$; or in a *sidereal* day mean time *loses* on sidereal time $3^m\ 55^s.9094$. In 1^h of sidereal time the loss is $9^s.8296$.

If t and s in the last term are expressed in *hours*, (85) and (86) become

$$\left.\begin{aligned} s &= t + 9^s.8565\ t, \\ t &= s - 9^s.8296\ s\,; \end{aligned}\right\} \qquad (87)$$

by which the reductions may be more readily calculated, when the tables are not at hand.

105. PROBLEM 31. *To convert mean time at a given place into sidereal time.*

Solution. Let

λ represent the longitude of the place, expressed in time, + when *west*,

T, the local *mean* time,

S, the corresponding *sidereal* time,

t, the interval from mean noon in *mean* time (differing from T only by omitting the day),

s, the same interval in *sidereal* time,

S_0, the sidereal time of mean noon at Greenwich,

S'_0, the sidereal time of mean noon at the place;

then, since λ expresses the Greenwich time of local noon, (Art. 92),

$$S'_0 = S_0 + .0027379\ \lambda\,;$$

evidently $S = s + S'_0$.

and by (86) $s = t + .0027379\ t\,;$ (88)

whence we have

$$S = t + S_0 + .0027379\ (\lambda + t). \qquad (89)$$

The Almanac (page II.) contains S_0 for each Greenwich mean noon, under the head "Sidereal Time." It should be taken out for the given astronomical day of the place; .0027379 λ is then the reduction for longitude, *additive* in *west* longitude, *subtractive* in *east*. It, as well as .0027379 t, the reduction to a sidereal interval, may be taken from the second part of Tab. II. of the Almanac, or from Tab. LI. (Bowd.); or either may be computed by (86) or first of (87).

From (89), then, we have the following rule:

To the local mean time add the sidereal time of Green-

wich mean noon of the given astronomical day, the reduction of this sidereal time for the longitude of the place, and the reduction of the hours, minutes, &c., of the mean time to a sidereal interval.

The astronomical (solar) day is usually retained. But if it be desirable to state the sidereal day, as well as the hours, &c., of the sidereal time, we prefix to S_0 the sidereal day at the instant of mean noon, which is the same as the astronomical day after the vernal equinox of each year; one day less before that date. At the instant of the vernal equinox the sidereal time and mean solar time coïncide. Before that time the mean sun transits before the vernal equinox; after that time it transits after the vernal equinox.

106. $T+\lambda$ is the Greenwich mean time. When this is given, or found in the course of computation, it will be more convenient to take out S_0 for the Greenwich day, and the combined reduction, .0027379 $(t+\lambda)$, for the hours, minutes, &c., of Greenwich mean time, instead of for t and λ separately.

It should be noted, however, that in the first method (Art. 105), S_0 is taken out for the local day; in this, it is taken out for the Greenwich day, provided $\lambda+t$, as used, expresses properly the Greenwich time.

107. $S_0+.0027379\ (t+\lambda)$ is the "sidereal time" of the Almanac interpolated for the Greenwich mean time. It is more convenient to term it the *right ascension of the mean sun* (Art. 93); and then the translation of (89) will be, *the sidereal time is equal to the right ascension of the mean sun + the mean time.*

This is also evident from Fig. 21, in which

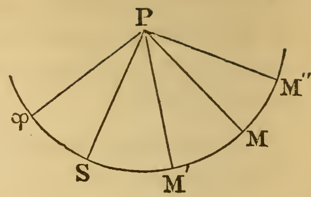

Fig. 21.

P is the pole;
P M, the meridian;
♈, the vernal equinox;
♈ M, the equator.
♈ M is also the *right ascension of the meridian*, and measures
M P ♈, the hour-angle of ♈, or the *sidereal* time (Art. 65).

If P S is the declination-circle passing through the mean sun,
♈ S is the right ascension of the mean sun, and
M P S is its hour-angle or the *mean* time (Art. 72), and is measured by the arc of the equator, S M.

Evidently $$♈ M = ♈ S + S M. \tag{90}$$

The hour-angles M P ♈, M P S, are reckoned from the meridian toward the west; hour-angles east from the meridian are then regarded as negative.

If P S is the declination-circle of the true sun, then will

♈ S be the right ascension, and
M P S the hour-angle of the *true* sun; and
S M will measure the *apparent* time,

and the interpretation of (90) will be, *the sidereal time is equal to the right ascension of the true sun + the apparent time.*

EXAMPLES.

1. Find the sidereal time of 1865, Jan. 30, $10^h\ 15^m\ 26^s.6$, ast. mean time in long. 150° 13′ 10″ ($10^h\ 0^m\ 52^s.7$) W.

First Method.

		h m s
L. m. t.	Jan. 30	10 15 26.6
S_0		20 38 56.00
Red. for long.		+ 1 38.71
Red. of L. m. t.		+ 1 41.10
Sid. t.		6 57 42.4

Second Method.

		h m s
L. m. t.	Jan. 30	10 15 26.6
Long.		+ 10 0 52.7
G. m. t.	Jan. 30	20 16 19.3
L. m. t.		10 15 26.6
S_0		20 38 56.00
Red. for G. m. t.		+ 3 19.81
Sid. t.		6 57 42.4

2. Find the sidereal time of 1865, Jan. 30, $10^h\ 15^m\ 26^s.6$, ast. mean time in long. $10^h\ 0^m\ 52^s.7$ E.

		h m s	
L. m. t.	Jan. 30	10 15 26.6	
S_0		20 36 56.00	
Red. for long.		— 1 38.71	2d part of Tab. II.
Red. of L. m. t.		+ 1 41.10	2d part of Tab. II.
Sid. t.		6 54 25.0	

3. Find the sidereal time of 1865, Sept. 25, $21^h\ 16^m\ 15^s$, in long. $60°\ 13'$ ($=4^h\ 0^m\ 52^s$) W.

		h m s
L. m. t.	Sept. 25	21 16 15
S_0		12 17 15.9
Red. for long.		+ 0 39.6
Red. of L. m. t.		+ 3 29.7
Sid. t.		9 37 40

		h m s
L. m. t.	Sept. 25	21 16 15
Long.		+ 4 0 52
G. m. t.	Sept. 26	1 17 7
S_0		12 21 12.7
Red. for G. m. t.		+ 12.6
Sid. t.		9 37 40

4. Find the sidereal time of 1865, Sept. 25, $3^h\ 16^m\ 15^s.0$, in long. $8^h\ 16^m\ 25^s.3$ E.

		h m s
L. m. t.	Sept. 25	3 16 15.0
S_0		12 17 15.89
Red. for long.		— 1 21.55
Red. for L. m. t.		+ 32.24
Sid. t.		15 32 41.6

		h m s
L. m. t.	Sept. 25	3 16 15.0
Long.		— 8 16 25.3
G. m. t.	Sept. 24	18 59 49.7
S_0		12 13 19.34
Red. for G. m. t.		+ 3 7.25
Sid. t.		15 32 41.6

108. PROBLEM 32. *To convert sidereal time at any place into mean time.*

1*st Solution.* The sidereal time at mean noon at the place is from (88)

$$S'_0 = S_0 + .0027379\ \lambda;$$

the sidereal interval from mean noon,

$$s = S - S'_0 = S - S_0 - .0027379\ \lambda; \tag{91}$$

and from (86) the corresponding mean time interval,

$$t = s - .0027304\ s. \tag{92}$$

The mean time T is completed by prefixing to t the astronomical day.

From (91) and (92) we have the following rule:

From the local sidereal time subtract the sidereal time of Greenwich mean noon of the given astronomical day and the reduction of this sidereal time for the longitude of the place; and from the sidereal interval thus obtained subtract the reduction to a mean time interval; and to the result prefix the giv n astronomical day.

The local sidereal time may be increased by 24^h if necessary. The reduction for longitude, $.0027379\ \lambda$, may be taken from the 2d part of Tab. II. of the Almanac, or from Tab. LI. (Bowd.); *numerically*, it is *subtractive* in *west* longitude, *additive* in *east*, as applied to the given sidereal time. The reduction of the sidereal interval, $.0027304\ s$, may be taken from the 1st part of Tab. II., or from Tab. LII. (Bowd.), and is always subtractive.

2*d Solution.* Let

M_0 represent the "mean time of the *preceding* sidereal 0^h" * at Greenwich;

M'_0, the "mean time of the *preceding* sidereal 0^h" at the place;

S, the interval from 0^h in *sidereal* time;

t, the same interval in *mean* time:

then, since λ will be the sidereal interval between the Greenwich and local sidereal 0^h (Art. 92),

$$M'_0 = M_0 - .0027304\ \lambda,$$

evidently, $$T = t + M'_0,$$

and by (86) $$t = S - .0027304\ S;$$

whence we have

$$T = S + M_0 - .0027304\ (\lambda + S). \qquad (93)$$

* It is equal to 24^h—the right ascension of the mean sun. In the British Almanac it is called "Mean time of transit of first point of Aries."

The Almanac (page III.) contains M_0 for the Greenwich sidereal 0^h on each *mean* day. The Almanac date of the *preceding* sidereal 0^h is generally the *same* as the local astronomical date when the sidereal time is *less* than the "sidereal time at mean noon" (page II.), but 1^d *less* when the sidereal time is *greater* than that at mean noon. The doubtful case is when the mean time is within 4^m of noon: the comparison must then be made with the sidereal time at the nearest local mean noon.

The reduction of M_0 to the local meridian is $-.0027304\ \lambda$, which may be taken from the 1st part of Tab. II., or from Tab. LII. (Bowd.) It is *subtractive* in *west* longitude, *additive* in *east*.

The reduction of the sidereal interval, $.0027304\ S$, may be taken from the same tables; it is always *subtractive*.

The combined reduction, $.0027304\ (\lambda+S)$, may be taken out for the Greenwich sidereal time, $(\lambda+S)$, instead of for λ and S separately; but with these precautions, that when $\lambda+S>24^h$, M_0 may be taken out for 1^d later than stated in the previous precept, and interpolated for the excess of $(\lambda+S)$ over 24^h; and when $(\lambda+S)$ is negative, to retain its negative character, or else take out M_0 for one day earlier.

3*d Solution.* From (89) we have

$$t = S-[S_0+.0027379\ (t+\lambda)], \tag{94}$$

so that, when the Greenwich mean time $(t+\lambda)$ is sufficiently known, we may find for it the right ascension of the mean sun, (Art. 107)

$$S_0+.0027379\ (t+\lambda),$$

and subtract it from the given sidereal time: or, *the mean time is equal to the sidereal time—the right ascension of the mean sun.* So also we have from Art. 107 the precept:—*the apparent time is equal to the sidereal time—the right ascension of the true sun.*

Examples.

1. 1865, Jan. 30 (ast. day), in long $10^h\ 0^m\ 52^s.7$ W., the sidereal time is $6^h\ 57^m\ 42.^s4$; find the mean time.

	h m s		h m s
L. sid. t.	6 57 42.4	L. sid. t.	6 57 42.4
— S_0 (Jan. 30)	−20 38 56.00	M_0 (Jan. 30)	3 20 31.06
— Red. for long.	— 1 38.71	Red. for long.	— 1 38.44
Sid. int.	10 17 7.69	Red. of sid. t.	— 1 8.43
Red. of sid. int.	— 1 41.10	L. m. t. Jan. 30	10 15 26.6
L. m. t. Jan. 30	10 15 26.6		

2. 1865, Jan. 30, (ast. day,) in long. $10^h\ 0^m\ 52^s.7$ E., the sidereal time is $6^h\ 54^m\ 25^s.0$; what is the mean time?

	h m s		h m s
L. sid. t.	6 54 25.0	L. sid. t.	6 54 25.0
— S_0 (Jan. 30)	−20 38 56.00	M_0 (Jan 30)	3 20 31.06
— Red. for long.	+ 1 38.71	Red. for long.	+ 1 38.44
Sid. int.	10 17 7.71	Red. of sid. t.	— 1 7.89
Red. of sid. int.	— 1 41.10	L. m. t. Jan. 30	10 15 26.6
L. m. t. Jan. 30	10 15 26.6		

3. 1865, Sept. 26, 9^h A.M., in long. $4^h\ 0^m\ 52^s$W., the sidereal time is $9^h\ 37^m\ 40^s.1$; find the mean time.

	h m s		h m s
L. sid. t.	9 37 40.1	L. sid. t.	9 37 40.1
— S_0 (Sept. 25)	−12 17 15.89	M_0 (Sept. 25)	11 40 48.98
— Red. for long.	— 39.57	Red. for long.	— 39.46
Sid. int.	21 19 44.64	Red. of sid. t.	−1 34.64
Red. of sid. int.	— 3 29.65	L. m. t. Sept. 25	21 16 15.0
L. m. t. Sept. 25	21 16 15.0		

4. 1865, Sept. 25, 3^h P.M., in long. $8^h\ 16^m\ 25^s.3$ E., the sidereal time is $15^h\ 32^m\ 41^s.6$; find the mean time.

	h m s		h m s
L. sid. t.	15 32 41.6	L. sid. t.	15 32 41.6
— S_0 (Sept. 25)	−12 17 15.89	M_0 (Sept. 24)	11 44 44.89
— Red. for long.	+ 1 21.55	Red. for long.	+ 1 21.33
Sid. int.	3 16 47.26	Red. of sid. t.	— 2 32.80
Red. of sid. int.	— 32.24	L. m. t. Sept. 25	3 16 15.0
L. m. t. Sept. 25	3 16 15.0		

RELATION OF HOUR-ANGLES AND TIME.

109. PROBLEM 33. *To find the mean time of meridian transit of a celestial body, the longitude of the place or the Greenwich time being known.*

Solution. In the case of the sun the instant of meridian transit is *apparent noon* of the place; for which we have (84)

$$T_m = E, \text{ the equation of time,}$$

which can be taken from page I. of the Almanac, and interpolated for the longitude, which in this case is also the Greenwich apparent time; or from page II., and interpolated for the Greenwich mean time. When E is subtractive, the subtraction from the number of days can be performed.

The apparent right ascension of any body at the instant of its meridian transit is also the right ascension of the meridian, or *sidereal time.* (Art. 65.) It suffices therefore to find the right ascension of the body, and, regarding it as the *sidereal* time, reduce it to *mean* time by Problem 31.

The American Ephemeris contains the apparent right ascensions of two hundred principal stars for the upper culminations at Washington ; the British Almanac contains the positions of one hundred for the upper culminations at Greenwich. They are reduced to any other meridian, when necessary, by interpolating for the longitude.

The right ascensions of the moon are given for each hour, and of the planets for each noon, of Greenwich mean time,

and may be found for a given Greenwich mean time by Problem 21. If, however, the longitude of the place is given, the local mean time of transit of the moon, or a planet, may first be found from the Almanac to the nearest minute or tenth (Probs. 25, 26); then for this mean time the right ascensions of the moon, or of the planet (Prob. 21), and of the mean sun (Prob. 24), may be computed. Subtracting the right ascension of the mean sun from the right ascension of the moon, or planet, will give the mean time of transit (Prob. 32, 3d solution). If it differ sensibly from that previously obtained, the process may be repeated with this new approximation.

If the time of transit has been noted by a clock, or chronometer, regulated either to local or Greenwich time, it should be used in preference to the approximate time of transit found from the Almanac in computing the right ascensions.

The American Ephemeris contains also the right ascensions of the moon and principal planets at their transits of the upper meridian at Washington. They can be reduced to any other meridian by interpolating for the longitude from Washington.

This solution will give the time of the upper culmination of a heavenly body. To find the time of a lower culmination, 12^h may be added to the right ascension of the body, if sufficiently well known; or, as is generally preferable, 12^h may be added to the longitude of the place. The instant of a lower culmination on any meridian will be that of an upper culmination on the opposite meridian.

Examples.

1. Find the times of meridian passage of the moon and Jupiter for 1865, June 6 (civil day), in long 100° 15′ W. (Example 1, Art. 95, p. 93.)

	☽		♃	
	h m		h m	
Approx. m. t.	June 6 10 12.9		June 5 12 40.6	
Long.	+6 41.0		+6 41.0	
G. m. t.	June 6 16 53.9		June 5 19 21.6	
	h m s	s	h m s	s
R. As'n.	15 13 28.54	+2.1088	17 39 52.80	— 1.340
Red. for G. m. t.	+ 1 53.66	105.440 6.326 1.898	— 25.64	13.40 11.760 .402 80
R. As'n at transit	15 15 22.20		17 39 27.16	
S_0	4 59 38.32		4 55 41.76	
Red. for G. m. t.	+ 2 46.56		+ 3 10.82	
S'_0	5 2 24.88		4 58 52.58	
M. t. of transit,	June 6 10 12 57.32		June 5 12 40 34.58	
Diff. from approx. t.	+ 3.32		— 1.42	
In $3^s.32$ Ch. of R. A.	+ .117			
In $3^s.32$ — Ch. of S_0	— .009			
M. t. of transit,	June 6 10 12 57.43			

110. Problem 34. *To find the hour-angle of the sun for a given place and time.*

Solution. The hour-angle of the sun, reckoned from the upper meridian toward the west, is the *apparent* time reckoned astronomically (Art. 72). Its hour-angle east of the meridian is negative, and numerically equal to 24^h—the apparent time.

A given *mean* or *sidereal* time must then be converted into *apparent* time; for this, the longitude, or the Greenwich time, must be known approximately.

111. Problem 35. *To find the hour-angle of the moon, a planet, or a fixed star, for a given place and time.*

Solution. In Fig. 21, as described in Art. 104,

♈ M is the right ascension of the meridian, and measures M P ♈, the *sidereal* time.

Let

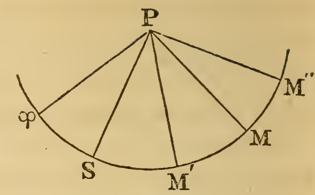

Fig. 21.

P S be the declination-circle of the *mean* sun, then

♈ S is the right ascension of the mean sun, and

M P S is the *mean* time, and is measured by the arc of the equator, S M.

Let

P M′ be the declination-circle of some other celestial body; then

♈ M′ is its right ascension, and

M P M′ is its hour-angle, and is measured by the arc M′ M.

From the figure,

$$M' M = ♈ M - ♈ M' = ♈ S + S M - ♈ M'. \qquad (95)$$

If ♈ S is the right ascension of the *true* sun,
S M will measure the *apparent* time.

From (95), then, we have the following rule:—

To a given *apparent* time add the right ascension of the *true* sun; or to a given *mean* time add the right ascension of the *mean* sun, to find the corresponding *sidereal* time. Then from the *sidereal* time subtract the body's right ascension; the difference is the hour-angle west from the meridian. If it is more than 12^h, it may be subtracted from 24^h: the hour-angle, then, is —, or east of the meridian. It is necessary to know the longitude, or the Greenwich time, sufficiently near to find the right ascensions of the sun and body.

112. Problem 36. *To find the local time, given the hour-angle of the sun and the Greenwich time.*

Solution. The hour-angle reckoned westward is itself the local *apparent* time, which may be reduced to *mean* or *sidereal* time (Probs. 29, 30), as may be required. The Green-

wich time, or the longitude of the place, is needed only for this reduction.

113. Problem 37. *To find the local time, given the hour-angle of some celestial body and the Greenwich time.*

Solution. Find from the Almanac for the Greenwich time (Prob. 21) the right ascension of the body. Then, from (95), we have

$$\Upsilon M = \Upsilon M' + M' M,$$

from which, and Arts. 105, 107, we have the following rule, regarding hour-angles to the east as negative :—

To the right ascension of the body add its hour-angle, the result is the *sidereal* time. From this subtracting the right ascension of the *true* sun gives the *apparent* time ; or the right ascension of the *mean* sun gives the *mean* time.

The Greenwich time is needed for finding the required right ascensions.

If the longitude of the place is given, but not the Greenwich time, we may first use an estimated Greenwich time, and then revise the computations with a corrected value, until the assumed and computed values sufficiently agree.

Examples.

1. 1865, Jan. 16, $12^h\ 15^m\ 17^s.6$, mean time in long. 150° 13′ 10″ W., find the hour-angle of the moon.

		h m s			h m s
L. m. t.	Jan. 16	12 15 17.6		L. m. t. Jan. 16	12 15 17.6
Long.		+ 10 0 52.7		S_0	19 43 44.22
G. m. t.	Jan. 16	22 16 10.3		Red. for long.	+ 1 38.71
☽'s R. A. (Jan. 16 22^h)		11 48 31.61	$+1^s.8584$	Red. of L. m. t.	+ 2 0.79
			18 .584		
Red. for G. m. t.		+ 30.05	11 .150	L. sid. t.	8 2 41.32
			.186		
			.133		
☽'s R. A. at date	. . .	. . .	. . .	. . .	11 49 1.66
				☽'s hour-angle	— 3 46 20.34

2. 1865, Jan. 16 $22^h\ 16^m\ 10^s.3$, G. mean time, the moon's hour-angle is $-3^h\ 46^m\ 20^s.3$; find the local mean time.

		h m s	
☽'s hour-angle		− 3 46 20.3	
☽'s R. A. (Jan. 16 22^h)		+11 48 31.61	$+1^s.8584$
			18 .584
Red. for G. m. t.		+ 30.05	11 .150
			.186
L. sid. t.		8 2 41.36	.133
$-S_0$	(Jan. 16)	−19 43 44.22	
—Red. for G. m. t.		— 3 39.50	
L. m. t.	Jan. 16	12 15 17.6	

Subtracting this from the G. m. t. gives for the longitude $10^h\ 0^m\ 52^s.7$ W.

3. 1865, Jan. 16, 12^h (nearly) in long. 150° 13′ 10″ W., the moon's hour-angle is $-3^h\ 46^m\ 20^s.3$; find the local mean time.

		h m s
	Long.	10 0 52.7
	☽'s h. ang.	$-3^h\ 46^m.3$
in $-3^h.8$	ch. of R. A.	− 7 .0
	—ch. of S_0	+ 0 .6

		h m	m
☽'s mer. pass.	Jan. 16	15 50.7	+1.74
Red. for long.		+17.4	
	Jan. 16	16 8.1	
. . . . .		− 3 52.7	
1st approx. L. m. t.	Jan. 16	12 15.4	
Long.		+10 0.8	
1st approx. G. m. t.	Jan. 16	22 16.2	

		h m s			
☽'s h. ang.		−3 46 20.3			
☽'s R. A. (Jan. 16 22^h)		+11 48 31.61	$+1^s.8584$		
			18 .584		
Red. for G. m. t.		+30.11	11 .150	ch. in $-1^s.6$	−.046
			.372		
L. sid. t.		8 2 41.42			
$-S_0$ (Jan. 16)		−19 43 44.22			
—Red. for G. m. t.		−3 39.50		—ch. in $-1^s.6$	+.004
2d L. m. t.	Jan. 16	12 15 17.70		cor. for $-1^s.6$	−.04
Long.		10 0 52.7			
2d G. m. t.	Jan. 22	22 16 10.4			
Diff. from 1st G. m. t.		−1.6			
3d L. m. t.	Jan. 16	12 15 17.7			

CHAPTER VI.

NAUTICAL ASTRONOMY.

ALTITUDES.—AZIMUTHS.—HOUR-ANGLES AND TIME.

115. NAUTICAL Astronomy comprises those problems of Spherical Astronomy which are used in determining geographical positions, or in finding the corrections of the instruments employed. In general, they admit of a much more refined application on shore, where more delicate and stable instruments can be used, than is possible at sea, where the instability of the waves and the uncertainty of the sea-horizon present practical obstacles, both to precision in observations and to the accuracy of the results, which cannot be obviated.

116. In the problems which are here discussed the following notation will be employed :—

$L =$ the latitude of the place of observation ;
$h =$ the true altitude of a celestial body ;
$z = 90° - h$, its zenith distance ;
$d =$ its declination ;
$\gamma =$ its polar distance ;
$t =$ its hour-angle ;
$Z =$ its azimuth.

Let the diagram (Fig. 22) represent the projection of the celestial sphere on the plane of the horizon of a place :—

Z, the zenith of the place ;
N Z S, its meridian ;

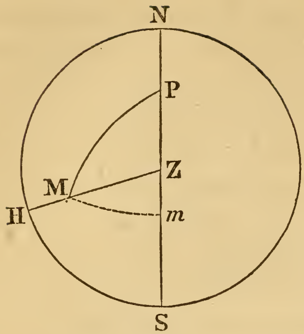

Fig. 22.

P, the elevated pole, or that whose name is the same as that of the latitude;

M, the position of a celestial body;

Z M H, a vertical circle; and

P M, a declination-circle, through M.

Then, in the spherical triangle P M Z,

P Z $= 90° - L$, the co-latitude of the place;
P M $= p = 90° - d$, the polar distance of M;
Z M $= 90° - h$, the complement of its altitude, or its zenith distance;
Z P M $= t$, its hour-angle;
P Z M $= Z$, its azimuth.

The angle P M Z is rarely used, but is sometimes called the *position angle* of the body.

This triangle, from its involving so many of the quantities which enter into astronomical problems, is called the *astronomical triangle.* As three of its parts are sufficient to determine the rest, if three of the five quantities L, d, h, t, and Z are known, the other two may be found by the usual formulas of spherical trigonometry. These admit, however, of modifications which better adapt them for practical use. The following articles point out how L, d, h, and t may be obtained.

117. The *latitudes* and *longitudes* of places on shore are given upon charts, but more accurately in tables of geographical positions, such as are found in books of sailing-directions, and in Tab. LIV. (Bowd.) At sea it is sometimes necessary to assume them from the dead reckoning brought forward from preceding, or carried back from subsequent, determinations. (Bowd., p. 264.)

118. The *altitude* of an object may be directly measured at sea above the sea-horizon with a quadrant or sextant; on shore, with a sextant and artificial horizon, or with an *altitude circle.* All measurements with instruments require correction for the errors of the instrument. Observed altitudes require reduction for refraction and parallax; for semidiameter, when a limb of the object is observed; and at sea, for the dip of the horizon. The reductions for dip and refraction are *subtractive;* for parallax, *additive.* Strictly, the reductions should be made in the following order: for *instrumental errors*, *dip*, *refraction*, *parallax*, *semidiameter.* In ordinary nautical practice it is unnecessary to observe this order.

Following it we should have,—

1st. The reading of the instrument with which an altitude is measured;

2d. The corrected reading or *observed* altitude of a limb;

3d. The *apparent* altitude of the limb;

4th. When corrected for refraction and parallax, the *true altitude* of the limb;

5th. The *true* altitude of the centre.

Except with the sea-horizon, the observed and apparent altitudes are the same. For the fixed stars, and for the planets when their semidiameters are not taken into account, the altitudes of the limb and the centre are the same.

Unless otherwise stated, the *true altitude of the centre* is the altitude which enters into the following problems, and is denoted by h.

119. The *hour-angle* of a body can be found, when the local time and longitude, or the Greenwich time, are given. (Probs. 34, 35.) For noting the time of an observation, a clock, chronometer, or watch is used; at sea, only the last two; but it will be necessary to know how much it is too fast or too slow of the particular time required.

120. The *declination* of a body can be found when the Greenwich time is known. (Prob. 21.)

The polar distance of a heavenly body is the arc of the declination-circle between the body and the elevated pole of the place; that is, the *north* pole, when the place is in *north* latitude; the *south* pole, when it is in *south* latitude. If

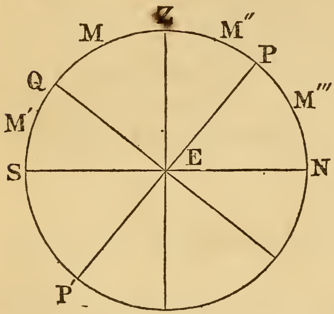

Fig. 23.

P P′ (Fig. 23) is the projection of the declination-circle through an object, M;

P, the north pole;

P′, the south pole;

E Q, the equator; then the polar distances,

$$P\,M = P\,Q - Q\,M = 90° - d,$$
$$P'M = P'Q + Q\,M = 90° + d.$$

That is, the polar distance is $90° - d$ or $90° + d$, according as the pole from which it is reckoned is N. or S. This, however, is regarding declination, like the latitude, as positive when N., negative when S.

To avoid, however, the double sign in the investigation of the formulas of Nautical Astronomy, we shall in most cases consider the declination, which is of the *same* name as the latitude, as *positive*, and that which is of a *different* name from the latitude, as *negative;* hence the polar distance will be represented by

$$p = 90° - d.$$

When the declination is of a different name from the latitude, we have *numerically*

$$p = 90° + d.$$

ALTITUDE AND AZIMUTH.

121. Problem 38. *To find the altitude and azimuth of a heavenly body at a given place and time.*

Solution. Find the declination of the body and its hour-angle at the given time. (Probs. 21, 34, and 35.)

Then in the spherical triangle P M Z (Fig. 24), we have given

$$P\,Z = 90° - L,$$
$$P\,M = 90° - d,$$
$$Z\,P\,M = t,$$

to find

$$Z\,M = 90° - h,$$
$$P\,Z\,M = Z.$$

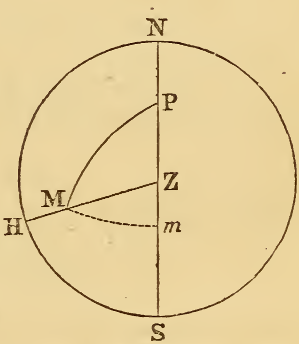

Fig. 24.

By Sph. Trig. (122), (123), if in the triangle A B C (Fig. 25), we have given b, c, and A to find a and B, we have

$$\left.\begin{aligned} \tan \phi &= \tan b \cos A, \\ \cos a &= \frac{\cos (c - \phi) \cos b}{\cos \phi}, \\ \cot B &= \frac{\sin (c - \phi) \cot A}{\sin \phi}, \end{aligned}\right\}$$

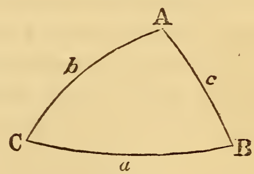

Fig. 25.

which, by substituting the corresponding parts of the triangle P Z M, give

$$\left.\begin{aligned} \tan \phi &= \cot d \cos t, \\ \sin h &= \frac{(\sin \phi + L) \sin d}{\cos \phi}, \\ \cot Z &= \frac{\cos (\phi + L) \cot t}{\sin \phi}. \end{aligned}\right\} \qquad (96)$$

If we put $\phi = 90° - \phi'$, these become

$$\left.\begin{aligned} \tan \phi' &= \tan d \sec t, \\ \sin h &= \frac{(\cos \phi' - L) \sin d}{\sin \phi'}, \\ \cot Z &= \frac{\sin (\phi' - L) \cot t}{\cos \phi'}, \end{aligned}\right\} \quad (97)$$

which afford the convenient precept, *ϕ' has the same name, or sign, as the declination, and is numerically in the same quadrant as t.*

122. When $t = 6^h$, $\phi' = 90°$, and the 3d of (97) assumes an indeterminate form. But from the 1st we have

$$\cot t = \frac{\tan d}{\tan \phi' \sin t};$$

which, substituted, gives

$$\cot Z = \frac{\sin (\phi' - L) \tan d}{\sin \phi' \sin t}, \quad (98)$$

which may be used when t is near 6^h.

123. h is the *true* altitude of M. If the *apparent* altitude is required, the parallax (Art. 54) must be subtracted, and the refraction (Art. 41) added.

Z is the *true* bearing, or azimuth, of the body, reckoned from the N. point of the horizon in *north* latitude, and from the S. point in *south* latitude. It is generally most convenient to reckon it as positive toward the *east*, which will require in the above formulas $-Z$ for Z, since t is positive when west. Restricting, however, Z numerically to 180°, it may be marked E. or W., like the hour-angle.

124. In Fig. 24, if Mm be drawn perpendicular to the meridian, then

P $m = \phi = 90° - \phi'$,
Z $m = (\phi + L) - 90° = L - \phi'$; or,
ϕ is the polar distance of m,
ϕ', its declination,

$L-\phi$, its zenith distance, positive, or of the same name as the latitude, toward the equator. A convenient precept is to mark it N. or S., according as the zenith is N. or S. of the point m.

m falls on the same side of the zenith as the equator when $Z > 90°$; at the zenith when $Z = 90°$; and on the same side as the elevated pole when $Z < 90°$. It falls between P and Z only when t and Z are both less than 90°.

125. In the case of α Ursæ Minoris (*Polaris*), whose polar distance is 1° 25′, the more convenient formulas derived from (96) will be, since p and ϕ are small,

$$\phi = p \cos t,$$

(which gives ϕ within 0″.5)

$$\sin h = \sin (L+\phi) \frac{\cos p}{\cos \phi},$$

$$\tan Z = \frac{\tan p \sin t \cos \phi}{\cos (L+\phi)};$$

or approximately,

$$h = L+\phi,$$
$$Z = p \sin t \sec (L+\phi).$$

Z is a maximum, or the star is at its greatest elongation, when the angle Z M P (Fig. 24), or Z n P (Fig. 30), is 90°. We then have

$$\sin Z = \sin p \sec L,$$

or nearly

$$Z = p \sec L.$$

126. Problem 39. *To find the altitude of a heavenly body at a given place and time, when its azimuth is not required.*

Solution. The 1st and 2d of (96) or (97) may be used; or, by Sph. Trig. (4),

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

we have $\sin h = \sin L \sin d + \cos L \cos d \cos t$;

which, since $\cos t = 1-2 \sin^2 \tfrac{1}{2} t,$

reduces to

$$\left.\begin{array}{ll} & \sin h = \cos (L-d) - 2 \cos L \cos d \sin^2 \tfrac{1}{2} t, \\ \text{or} & \sin h = \cos (L-d) - \cos L \cos d \text{ versin } t. \end{array}\right\} \qquad (99)$$

$(L-d)$ becomes *numerically* $(L+d)$ when L and d are of different names.

Tab. XXVII. contains for the argument t in column P.M. the log sin $\frac{1}{2} t$ in the column of *sines;* which, doubled, is log $\sin^2 \frac{1}{2} t$. It is well to note this, for mistakes are often made by regarding the logarithms in this table as log sin, log cos, &c., of t instead of $\frac{1}{2} t$.

Tab. XXIII. contains for the argument t, log $2 \sin^2 \frac{1}{2} t =$ log versin t, with the index increased by 5.

It is sometimes necessary to compute the altitude of one, or both bodies, to use in connection with an observed lunar distance. The rules for this purpose on pp. 247, &c., Bowd., are derived from the above formulas. The result is evidently more accurate, the smaller the hour-angle t, especially if the altitude is near 90°. In these rules it is best to find the "sidereal time," or "right ascension of the meridian," from the *mean* local time, instead of the *apparent* (Art. 105).

127. Problem 40. *To find the azimuth of a heavenly body from its observed altitude at a given place.*

Solution. In this the Greenwich time of the observation must be known sufficiently near for finding the declination of the body. The observed altitude must be reduced to the true altitude. Then in the triangle P Z M we have given the three sides to find the angle P Z M.

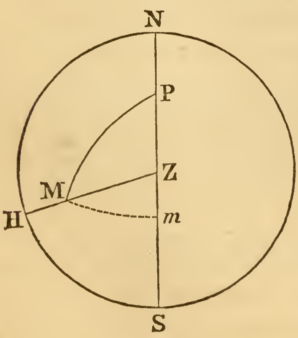

Fig. 26.

In the triangle A B C, putting $s = \frac{1}{2}(a+b+c)$, we have

$$\cos \tfrac{1}{2} \mathrm{B} = \sqrt{\left(\frac{\sin s \sin (s-b)}{\sin a \sin c}\right)}.$$

For the triangle P Z M,

$$
\begin{aligned}
B = Z, \qquad a &= 90^\circ - h, \; h \text{ being the true altitude,} \\
b &= p, \qquad \text{the polar distance,} \\
c &= 90^\circ - L, \text{ the co-latitude,} \\
s &= 90^\circ - \tfrac{1}{2}\,(L + h - p), \\
s - b &= 90^\circ - \tfrac{1}{2}\,(L + h + p),
\end{aligned}
$$

and the formula becomes

$$\cos \tfrac{1}{2}\, Z = \sqrt{\left(\frac{\cos \frac{1}{2}\,(L+h+p) \cos \frac{1}{2}\,(L+h-p)}{\cos L \cos h}\right)};$$

or, if we put

$$\left.\begin{aligned} s' &= \tfrac{1}{2}\,(L+h+p), \\ \cos \tfrac{1}{2}\, Z &= \sqrt{\left(\frac{\cos s' \cos (s'-p)}{\cos L \cos h}\right)}. \end{aligned}\right\} \qquad (100)$$

which accords with Bowditch's rule, p. 160.

In a similar way we may find from the formula

$$\sin \tfrac{1}{2}\, B = \sqrt{\left(\frac{\sin (s-a) \sin (s-b)}{\sin a \sin c}\right)},$$

$$\sin \tfrac{1}{2}\, Z = \sqrt{\left(\frac{\cos \frac{1}{2}\,(\text{co } L+h+d) \sin \frac{1}{2}\,(\text{co } L+h-d)}{\cos L \cos h}\right)},$$

in which $\qquad \text{co } L = 90^\circ - L;$

or, if we put

$$\left.\begin{aligned} s'' &= \tfrac{1}{2}\,(\text{co } L+h+d), \\ \sin \tfrac{1}{2}\, Z &= \sqrt{\left(\frac{\cos s'' \sin (s''-d)}{\cos L \cos h}\right)}. \end{aligned}\right\} \qquad (101)$$

(100) is preferred when $Z > 90^\circ$; (101), when $Z < 90^\circ$.

If the body is in the visible horizon, then nearly

$$h = -\,(33' + \text{the dip}).$$

128. If the bearing of the body is observed with a compass at the same time that its altitude is measured, or if the bearing is observed and the local time noted, the *declination*, or *variation*, of the compass can be found. For, the true azimuth, or bearing, of the body can be found from its altitude (Prob. 40), or from the local time (Prob. 38); and the magnetic declination is simply the difference of the *true* and

magnetic bearings of the same object, determined simultaneously if the object is in motion. It is marked *E.* when the true bearing is to the *right* of the magnetic bearing, *W.* when the true bearing is to the *left* of the magnetic bearing. (Bowd., p. 161.)

129. The *amplitude* of a star when in the true horizon is its distance from the east or the west point, and is marked N. or S., according as it is north or south of that point. It is, therefore, the complement of the azimuth.

PROBLEM 41. *To find the amplitude of a heavenly body when in the horizon of a given place.*

Solution. Let the body be in the horizon at M (Fig. 27), $A =$ W M, its amplitude. The triangle P M N is right angled at N, and there are given

$$\text{P N} = L,$$
$$\text{P M} = 90° - d,$$

to find

$$\text{N M} = Z = 90° - A.$$

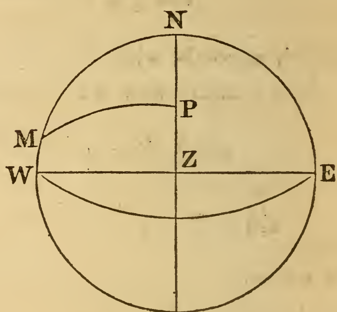

Fig. 27.

We have $\quad \cos \text{P M} = \cos \text{P N} \cos \text{N M},$

or $\quad \sin d = \cos L \cos Z,$

whence $\quad \cos Z = \sin A = \sin d \sec L, \qquad (102)$

as in Bowditch, p. 159. By (102) A is N or S like the declination.

As the equator intersects the horizon of any place in the east or west points, it is plain that the star will rise and set *north* or *south* of these points, according as its declination is N. or S.

Tab. VII. (Bowd.) contains the amplitude, A, for each 1° of latitude up to 66°, and each 1° of declination to 23°. The convenience of this table, in the case of the sun, is the only

reason for introducing amplitudes. It is generally best to express the bearing of an object by its azimuth.

In this problem the body is supposed to be in the true horizon, or about (33′ + the dip) above the visible horizon. Hence the rule to "observe the bearing of the sun, when its centre is about one of its diameters above the visible horizon." (Bowd., p. 158.)

Examples. (Probs. 38—41.)

1. 1865, Jan. 25, $2^h\ 33^m\ 13^s$ local mean time in lat. 49° 30′ S. long. 102° 39′ 15″ E.; required the sun's true altitude and azimuth. (97)

	h m s					
L. m. t. Jan. 25	2 33 13	(Jan. 25.) ⊙'s *dec.*			*Eq. of t.*	
Long.	−6 50 37	18° 52′ 48″.7 S.	−37″.35		$-12^m\ 41^s.67$	$-0^s.561$
			149 .4			2 .24
G. m. t. Jan. 24	19 42 36	+2 40 .3	7 .5		+2 .40	.11
or Jan. 25	$-4^h.39$	18 55 29 S.	3 .4		−12 39 .3	5
Eq'n of t.	−12 39.3					
L. ap. t.	2 20 33.7					

$t =$	35° 8′ 25″	l. sec 0.08738		l. cot 0.15251
$d =$	18 55 29 S.	l. tan 9.53512	l. sin 9.51098	
$\phi' =$	22 44 50 S.	l. tan 9.62250	l. cosec 0.41266	l. sec 0.03516
$L =$	49 30 S.			
$L-\varphi' =$	26 45 10 N.		l. cos 9.95083	l. sin 9.65335 *n*
$h =$	48 30 6		l. sin 9.87447	
$Z =$	S 124 44 23 W.			l. cot 9.84102 *n*

The reduction for refraction and parallax of $h = 48°.5$ is +45″; and the apparent altitude is $h' = 48°\ 30'\ 51''$. If the compass bearing of the sun at the same instant had been N. 34° 20′ W. = S. 145° 40′ W., the magnetic declination would have been 20° 56′ W.

2. 1865, Sept. 27, $5^h\ 20^m\ 16^s$ A.M. mean time in lat. 50° 15′ N., long. 87° 30′ W.; required the altitude and azimuth of Venus. (97)

L. m. t. Sept. 26	17h 20m 16s	Long.	+5h 50m 0s			
S_0	12 21 12.4	G. m. t. Sept. 26	23 10 16	=Sept. 27—0h.829		
Red. for long.	+57.5	(Sept. 27) *R.A.*			*Dec.* of Venus.	
Red. for L. m. t.	+2 50.9	10h 4m 14.13s	+11.645s		+12° 32′ 25.0″	−55.42″
			9.32			44.3
L. sid. t.	5 45 16.8	−9.65	.23		+45.9	1.1
R. A. of ♀	10 4 4.5	. . .	.10		+12 33 11	.5
H. A. of ♀	−4 18 47.7					

$t=$	64° 41′ 55″ E.	l. sec 0.36919		l. cot 9.67461
$d=$	12 33 11 N.	l. tan 9.34766	l. sin 9.33714	
$\phi'=$	27 31 11 N.	l. tan 9.71685	l. cosec 0.33530	l. sec 0.05215
$L=$	50 15 N.			
$\phi'-L=$	22 43 49 S.		l. cos 9.96489	l. sin 9.58703 *n*
$h=$	25 42 42		l. sin 9.63733	
$Z=$	N 101 38 17 E.			l. cot 9.31379 *n*

3. 1865, July 20, 5^h 58^m 20^s A.M., mean time in lat. 38° 19′ 20″ N., long. 150° 15′ 30″ E.; required the sun's azimuth. (98)

L. m. t. July 19	17h 58m 20s	(July 19) ☉'s *dec.*		*Eq'n of t.*		
Long.	−10 1 2	+20° 49′ 8″.7	−27″.60	−5m 58s.34	−0s.165	
			193.2		1.155	
G. m. t. July 19	7 57 18=7h.955	−3 39.5	24.8	−1.31	.150	
			1.4		9	
Eq'n of t.	−5 59.6	+20 45 29	.1	−5 59 65		
L. ap. t. July 19	17 52 20.4					

$t=$	91° 54′ 54″ E.	l. cosec 1.47604 *n*	l. cosec 0.00024
$d=$	20 45 29 N.	l. tan 9.57868	l. tan 9.57868
$\phi'=$	95 2 17 N.	l. tan 1.05472 *n*	l. cosec 0.00168
$L=$	38 19 20 N.		l. sin 9.92219
$\phi'-L=$	56 42 57 N.		l. cot 9.50279
$Z=$	N 72 20 43 E.		

4. Required the apparent altitudes of the sun and moon, Nov. 9, 1865, about 9 A.M., in lat. 18° 25′ S., long. 84° 6′ W.; time by chro. 2^h 25^m 10^s; chro. slow of G. m. t. 10^m 15^s. (99)

T. by chro.	$2^h\ 25^m\ 10^s$	(Nov. 9) ⊙'s *dec.*		*Eq'n of t.*
Chro. cor.	+10 15	$16^\circ\ 57'\ 42''.5$ S. $+42''.67$		$+16^m\ 0^s.76$ $-0^s.230$
G. m. t. Nov. 9	2 35 25 $=2^h.590$	+1 50.5	85.34, 21.33, 3.84	− .60, +16 0.16 ; .460, .115, 21
−Long.	−5 36 24			
L. m. t. Nov. 8	20 59 1	$d=16\ 59\ 33$ S.	l. cos	9.98061
Eq'n of t.	+16 0.2	$L=18\ 25$ S.	l. cos	9.97717
L. ap. t. Nov. 8	21 15 1.2	$t=-2^h\ 44^m\ 58^s.8$	l. $\sin^2 \frac{1}{2} t$	9.09360 *
$L-d$	$= 1^\circ\ 25'\ 27''$ S.	cos .99969	log 2	0.30103
		−.22511	log	9.35241
⊙'s true alt.	= 50 46 0	sin .77458		
Ref. and par.	+40			* l. versin t = 9.39463
⊙'s ap. alt.	= 50 46 40			

L. m. t. Nov. 8	$20^h\ 59^m\ 1^s$	(Nov. 9 2^h) ☽'s *dec.*		☽'s *R. A.*
S_0	15 10 44.2	$13^\circ\ 4'\ 38''.9$ N. $-7''.420$		$8^h\ 42^m\ 28^s.70$ $+2^s.1280$
Red. for λ	+55.3		222.60	63.84
Red of t_m	+3 26.8	−4 22.8	37.10, 2.97, .12	+1 15.36, 8 43 44.1 ; 10.64, .88
L. sid. t.	12 14 7.3			
☽'s R. A.	8 43 44.1	$d=13\ 0\ 16$ N.	l. cos	9.98871
$t=$	3 30 23.2	$L=18\ 25$ S.	l. cos	9.97717
$L-d=$	$31^\circ\ 25'\ 16''$ S.		l. $\sin^2 \frac{1}{2}$ t	9.29290 *
		cos .85336	log 2	0.30103
		−.36292	log	9.55981
☽'s true alt.	29 22 10	sin .49044		
Par. and ref.	−49	Tab. XXIX.		* l. versin t = 9.59393
☽'s app. alt.	28 33 (approx.)	☽'s H. pa'x $56'\ 28''.3$ $-2''.10$		
		−5 .5	4 .2, 1 .3	
Par. and ref.	−47 47	$\pi=56\ 22\ .8$	log	3.5293
☽'s app. alt.	28 34 23	$h'=28^\circ\ 33'$	l. sin	9.9437
		$p=49'\ 32''$	log	3.4730
		ref. = −145		
		Par. and ref. = 47 47		
		By Tab. XIX. $47'\ 48''$		

5. Find the altitude and azimuth of Polaris, 1865, Sept. 25 $8^h\ 15^m$ P.M., in lat. $49^\circ\ 16'$ N., long. $85^\circ\ 16'$ W., (Art. 125).

	h m s					
L. m. t.	8 15 0	Long. $5^h\ 41^m\ 4^s$				
S_0	12 17 16	$p =$ 1° 24′ 27″	log	3.7048	l. cos	9.99987
Red. for long.	+ 56	$t =$ 69 8 0	l. cos	9.5517		
Red. of L. m. t.	+ 1 21	$\phi =$ 0 30 5	log	3.2565	l. sec	0.00002
L. sid. t.	20 34 33	$L+\phi =$ 49 46 5	l. sec	0.1898	l. sin	9.88277
R. A. of ∗	1 11 5	$h =$ 49 45 0	log p.	3.7048	l. sin h	9.88266
H. ang. of ∗	— 4 36 32		l. sin t.	9.9705		
		$Z =$ N. 2 2 10 E.	log	3.8651		

6. Required the greatest elongation of Polaris, 1865, Sept. 25, in lat 49° 16′ N.

$p =$ 1° 24′ 27″	log	3.7048	or,	l. sin	8.3903
$L =$ 49 16	l. sec	0.1854		l. sec	0.1854
$Z =$ 2 9 26	log	3.8902		l. sin	8.5757

7. At sea, 1865, May 20, $15^h\ 23^m\ 16^s$ mean time Greenwich, in lat. 40° 15′ S., long. 107° 15′ W., the observed altitude of the sun's lower limb 10° 15′ 20″, index correction of sextant + 3′ 20″, height of eye 18 feet, bearing of sun by compass N. 41° 45′ E.; required the sun's azimuth and the magnetic declination or variation. (100.)

G. m. t. May 20 $15^h\ 23^m\ 16^s = 15^h.388$ ☉'s dec. 20° 2′ 28″.0 N. + 30″.79

☉̲	10° 15′ 20″	In. cor. + 3′ 20″	+ 7 53.7	307.9
	+ 14 59	Dip. — 4 11		153.9
		S. diam. + 15 50	20 10 22 N.	9.2
				2.7
$h =$	10 30 19	l. sec 0.00734		
$L =$	40 15	l. sec 0.11734		
$p =$	110 10 22			
$2\,s =$	160 55 41			
$s =$	80 27 50	l. cos 9.21925		
$s - p =$	— 29 42 32	l. cos 9.93880		
		19.28273		
$\frac{1}{2} Z =$	64 1 48	l. cos 9.64137		
true $Z =$	S. 128 3 36 E.	= N. 51° 56′ 24″ E.		
Mag.		N. 41 45 E.		
Variation		10 11 E.		

8. 1865, Sept. 20, in lat. 30° 25′ N., long. 50° 16′ W., the compass bearing of the sun, when one of its diameters above the horizon, was S. 79° 30′ W.; required its true bearing and the variation. (102.)

L. ap. t. Sept. 20 6^h 0^m (Tab. IX.)	☉'s dec.	0° 59′ 32″	N.	— 58″.4
Long. + 3 21				525. 6
		— 9 6		17. 5
G. ap. t. Sept. 20 9 21 = $9^h.35$				2. 9
	$d=$	0 50	N.	l. sin 8.1627
	$L=$	30 25	N.	l. sec 0.0643
(Tab. VII.) $A=$ W. 0° 58′ N.	true $Z=$	N. 89 2	W.	l. cos 8.2270
	mag.	N. 100 30	W.	
	var.	11 28	W.	

9. On the same day at the same place, when the sun's centre was in the visible horizon, its compass bearing was S. 79° 30′ W.; height of eye 20 feet.

$p=$	89° 10	
$L=$	30 25	l. sec 0.0643
$h= -$	0 37	l. sec 0
$2s=$	118 58	
$s=$	59 29	l. cos 9.7057
$s-p= -$	29 41	l. cos 9.9389
		19.7089
$\frac{1}{2}Z=$	44 20	l. cos 9.8545
true $Z=$	N. 88 40 W.	
mag.	N. 100 30 W.	
var.	11 50 W.	

HOUR-ANGLE AND LOCAL TIME.

130. Problem 42. *To find the hour-angle of a heavenly body in the horizon.*

Solution. In the diagram of the last problem,

$$\text{M P Z} = t, \text{ the hour-angle;}$$

and in the triangle P M N are given

$$\left.\begin{array}{l} \text{P N} = L, \\ \text{P M} = 90° - d, \end{array}\right\} \text{to find} \quad \text{M P N} = 180° - t.$$

We have $$\cos M P N = \frac{\tan P N}{\tan P M},$$

whence $$\cos t = -\tan d \tan L. \tag{103}$$

131. From this it is apparent that when the latitude and declination have the same name, $t > 6^h$, and consequently that $2t$, or the time that the body is above the true horizon, $> 12^h$; and when the latitude and declination are of different names, $t < 6^h$ and $2t < 12^h$.

$2t$ is an interval of *sidereal* time for a fixed star, of *apparent* time for the sun.

In the case of the sun, t would be the apparent time of sunset, were the refraction and dip nothing, and $(24^h - t)$ would be the apparent time of sunrise.

Tab. IX. (Bowd.) contains t for different values of L and d.

132. PROBLEM 43. *To find the hour-angle of a heavenly body at a given place, and thence the local time, when the altitude of the body and the Greenwich time are known.*

Solution. Find the declination of the body for the Greenwich time, and reduce the observed altitude to the true altitude. Then in the triangle P Z M (Fig. 28) are given

$$P Z = 90° - L,$$
$$P M = p,$$
$$Z M = 90° - h,$$

to find

$$Z P M = t.$$

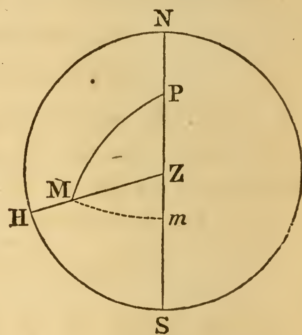

Fig. 28.

For the triangle A B C (Fig. 29), we have

$$\sin \tfrac{1}{2} A = \sqrt{\left(\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}\right)},$$

in which, putting $A = t$

$$a = 90° - h,$$
$$b = p,$$
$$c = 90° - L,$$

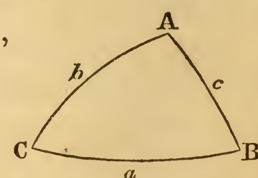

Fig. 29.

we have $s-b = 90°-\frac{1}{2}(L+p+h)$,

$s-c = \frac{1}{2}(L+p-h)$,

and

$$\sin \tfrac{1}{2} t = \sqrt{\left(\frac{\cos \frac{1}{2}(L+p+h) \sin \frac{1}{2}(L+p-h)}{\cos L \sin p}\right)};$$

or, if we put

$$\left.\begin{aligned} s' &= \tfrac{1}{2}(L+p+h), \\ \sin \tfrac{1}{2} t &= \sqrt{\left(\frac{\cos s' \sin (s'-h)}{\cos L \sin p}\right)}, \end{aligned}\right\} \qquad (104)$$

which is Bowditch's rule, p. 209.

From Tab. XXVII. (Bowd.) we may take t directly from column P. M., corresponding to the log sin $\frac{1}{2}$ t.

t is $-$ when the body is east of the meridian.

When the object is the sun west of the meridian, t is the apparent solar time; when the sun east of the meridian, $(24^h - t)$ is numerically the apparent time.

When the object is the moon, a planet, or a star, we have (Prob. 37), denoting its R. A. by a,

the *sidereal time* $= a+t$,

and the *mean time* $= a-S'_0+t$,

in which S'_0 is the "right ascension of the mean sun." (Art. 93.) Or the sidereal time may be converted into mean time by one of the other methods of Problem 32.

133. By the formula

$$\cos \tfrac{1}{2} A = \sqrt{\left(\frac{\sin s \sin (s-a)}{\sin b \sin c}\right)},$$

we may obtain for the triangle P Z M (z being the zenith distance),

$$\cos \tfrac{1}{2} t = \sqrt{\left(\frac{\sin \frac{1}{2}(\text{co } L+p+z) \sin \frac{1}{2}(\text{co } L+p-z)}{\cos L \sin p}\right)},$$

or putting

$$\left.\begin{aligned} s &= \tfrac{1}{2}(\text{co } L+p+z), \\ \cos \tfrac{1}{2} t &= \sqrt{\left(\frac{\sin s \sin (s-z)}{\cos L \sin p}\right)}, \end{aligned}\right\} \qquad (105)$$

which is the rule in Bowditch's 2d Method, p. 210.

(105) is preferable to (104) when t considerably exceeds 6^h, which may be the case in high latitudes.

If $L = 90°$, the horizon and equator coïncide, and $p + h = 90°$ and $p = z$; so that both (104) and (105) become indeterminate. In very high latitudes, then, these equations approach the indeterminate form, and it is impracticable to find with precision the local time from an observed altitude.

So also if $d = 90°$, the star is at the pole and $L = h$; and the problem is indeterminate. A great declination is therefore unfavorable.

134. If the object is in the visible horizon (rising or setting), $h = -(33' + \text{dip})$ nearly. With the sun, the instants when its upper and lower limbs are in the horizon may be noted, and the mean of the two times taken as the time of rising or setting of its centre. The irregularities of refraction would affect nearly alike the dip and the apparent position of the sun.

135. If the time at which the altitude is observed is noted by a watch, clock, or chronometer, we may readily find how much the watch or chronometer is fast or slow of the local time. (Prob. 50.) For, let

C be the time noted,

T, the local time deduced from the observation:

$c = T - C$ will be the *correction* of the watch or chronometer to reduce it to *apparent* time, when T is the local *apparent* time; to *mean* time, when T is the local *mean* time; or to *sidereal* time, when T is the local *sidereal* time.

136. The observed altitude is affected by errors of observation, errors of the instrument, and errors arising from the circumstances in which the observation is made; such as irregularities of refraction affecting both the position of the body and the dip of the horizon. Errors of the first

class are diminished by taking a number of observations. Thus several altitudes may be observed, and the time of each noted; and the mean of the altitudes taken as corresponding to the mean of the times, so far as the rate at which the body is rising or falling can be regarded as uniform during the period of observation. This period should then be brief.

137. We may easily find how much a supposed error of $1'$ in the altitude will affect the resulting hour-angle, by dividing the difference of two of the noted times by the difference in *minutes* of the two corresponding altitudes.

The effect will evidently be least when the body is rising or falling most rapidly. This will be the case when its diurnal circle makes the smallest angle with the vertical circle. An inspection of the diagram (Fig. 30) shows that this is the case when the object is nearest the prime vertical, or bears most nearly *east* or *west.*

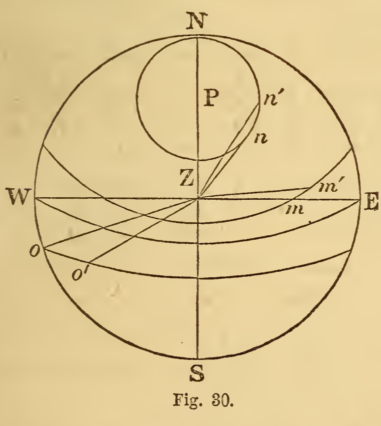

Fig. 30.

Thus $Z\,n$ being tangent to the diurnal circle $n\,n'$, the angle which it makes with it is 0; and is therefore less than the angle which any other vertical circle, as $Z\,n'$, makes with $n\,n'$.

The diurnal circle $m\,m'$ makes a smaller angle with $Z\,m$, the prime vertical, than with any other vertical circle, as $Z\,m'$.

The diurnal circle $o\,o'$ makes a smaller angle with $Z\,o$ than with $Z\,o'$.

The diurnal circles make right angles with the meridian; so that at the instant of transit, the change of altitude is 0.

138. At sea, and to a less extent on the land, the latitude is uncertain. To ascertain the effect of an error of 1′ in the assumed latitude, the hour-angles may be found for two latitudes separately, differing, say, 10′; and the difference of these hour-angles divided by 10.

This is an essential feature of Sumner's method, which will be explained hereafter. This method will also show that an error in latitude least affects the deduced hour-angle when the body is nearest the prime vertical.

Examples. (Prob. 43.)

1. At sea, 1865, March 20, $10^h\ 15^m\ 20^s$ G. mean time, in lat. 41° 15′ S., long. 86° 45′ W. (by account); observed P. M. altitude of the sun's lower limb 18° 20′; index. cor. of sextant −8′ 20″; height of eye 18 feet; required the local mean time. (104.)

G. m. t. Mar. 20	$10^h\ 15^m\ 20^s$	$\odot$'s *dec.*		*Eq'n of t.*	
	10.256	$-0°\ 2'\ 3''.7$	$+\ 59''.23$	$+7^m\ 33^s.82$	$-0^s.754$
			592.3		7.54
		$+\ 10\ \ 7.4$	11.8	$+\ 7.73$	.15
			3.3		4
		$+0\ \ 8\ \ 4$		$+7\ 26.1$	

$\underline{\odot}$	18° 20′ 0″	S. diam + 16′ 5	In. cor. —8′ 20″
	+ 52	par. + 8″	dip. —4 11
$h=$	18 20 52		ref. —2 50
$L=$	41 15	l. sec 0.12387	
$p=$	90 8 4	l. cosec 0	
$2s=$	149 43 56		
$S=$	74 51 58	l. cos 9.41677	
$S-h=$	56 31 6	l. sin 9.92120	
		19.46184	
L. ap. t. Mar. 20	$4^h\ 20^m\ 28^s$	l. sin ½ 9.73092	
Eq. of t.	+ 7 26		
L. m. t. Mar. 20	4 27 54		

Subtracting the local mean time from the G. mean time gives the long. $+\ 5^h\ 47^m\ 26^s = 86°\ 51\frac{1}{2}'$ W. If we take

$L = 41° 25'$ S., we shall find the local ap. time $4^h\ 20^m\ 12^s$; so that for $\Delta L = 10'$ S, $\Delta t = -16^s$.

2. 1865, Jan. 1, 21^h at the Navy-Yard, Havana, in lat. $23° 8' 39''$ N., long. $5^h\ 29^m\ 27^s$ W., the following altitudes of the sun were observed with an artificial horizon; required the local mean time.

	T. by Chro.	2 ⊙			
	h m s	° ′ ″		m s	
	3 33 57.5	58 10	Chro. fast of G. m. t.	42 37.7	
	34 29.3	20	Bar. 30.43		
	35 2.3	30	Ther. 75°		
	35 33.3	40	⊙'s diam. off the arc	+32′ 18″.3	
	36 4.7	50	⊙'s diam. on the arc	−32 35 .0	
	36 37.0	60	Index cor.	— 8. 3	
T. by Chro.	3 35 17.35	58 35	⊙'s *dec.*		
Chro. cor.	−42 37.7	h' = 29 17 30	½ In. cor. — 4″	−22° 53′ 42.2″	+14″.04
G. m. t. Jan. 2	3 52 39.6 = 3h.878	+14 42	S. diam. +16′ 18		42.12
			ref. — 1 40	+54.4	11.23
			par. +8	−22 52 48	.98
					.11
		h = 29 32 12		*Eq. of t.*	
				m s	s
		L = 23 8 29	l. sec 0.0364304	+4 27.81	+1.165
		p = 112 52 48	l. cosec 0.4102711		3.49
		$2s$ = 165 33 29		+4.51	.93
					8
		s = 82 46 45	l. cos 9.0993144	+4 32.32	1
		$s-h$ = 53 14 33	l. sin 9.9037276		
			19.4497435		
		$\frac{1}{2}t$ = −32° 3′ 16.7″	l. sin 9.7248718		
	h m s				
L. ap. t. Jan. 1	19 41 53.7	t = −64 6 33.4 = +295° 53′ 26″.6			
Eq. of t.	+ 4 32.3				
L. m. t. Jan. 1	19 46 26.0				

We have also by subtracting the chro. time from the local mean time,

Chro. cor. (L. m. t.)	$-6^h\ 12^m\ 4^s.8$
Long.	$+5\ \ 29\ \ 27.0$
Chro. cor. (G. m. t.)	$-0\ \ 42\ \ 37.8$

As the Chro. is fast, the *correction* is subtractive.

By comparing the first and last altitudes and the corresponding times, we find that for

$$2\,\Delta h = +50',\ \Delta t = +2^m\,39^s.5;\ \text{or, for}\ 2\,\Delta h = +1',\ \Delta t = +3^s.19;$$

that is, an error of 1′ in the double altitude will produce an error of 3^s in the resulting time.

3. At sea, 1865, Sept. 7, $8^h\ 4^m\ 16^s$, G. mean time, in lat. 46° 16′ N., long. 153° 0′ E., the observed altitude of the moon's upper limb, W. of the meridian, was 21° 19′; index cor. of octant, −3′; height of eye 20 feet; required the local mean time.

		h m		° ′		
G. m. t.	Sept. 7	8 4 16	☽	21 19	☽'s *dec.*	
			In. cor.	— 3	+6° 59′ 34″	+11″.2
			dip.	— 4	+48	44 .8
			S. diam.	— 17	+7 0 22	2 .8
			h' =	20 55	S. diam. 16′ 36″+4″	
			par.& ref.	+ 54	H. par. 60 49	
			h =	21 49		
			L =	46 16	l. sec	0.16033
			p =	83 0	l. cosec	0.00325
	h m s	s	$2s$ =	151 5		
☽'s R. A.	1 1 12.2	+2.40	s =	75 32½	l. cos	9.39738
	+ 10.2	9.6	$s-p$ =	53 43½	l. sin	9.90644
	1 1 22	.6				19.46740
☽'s H. A.	4 22 21	. . .	. . .	. . .	l. sin ½	9.73370
L. sid. t.	5 23 43					
$-S_0$	—11 6 18					
—Red. for G.m.t.	— 1 20					
L. m. t. Sept. 7	18 16 5					
Long.	— 10 11 49 = 152° 57′ E.					

4. 1865, Sept. 30, in lat 30° 27′ N., the Chro. time of the setting of the sun's centre was $11^h\ 16^m\ 6^s$; the Chro. cor., $+15^m\ 25^s$; height of eye 16 feet; required the local time.

	h m s	☉'s. dec.		*Eq'n of t.*		
T. by Chro.	11 16 6	° ′ ″	″	m s	s	
Chro. cor.	+ 15 25	— 2 54 31.9	—58.30	—10 4.40	—0.801	
			641.3		8.81	
G. m. t. Sept. 30	11 31 31	— 11 11.9	29.1	—9.23	.40	
	11.525	— 3 5 44	1.5	—10 13.6	2	

$h = -$	0° 37′		
$L =$	30 27	l. sec	0.06446
$p =$	93 6	l. cosec	0.00064
$2s =$	122 56		
$s =$	61 28	l. cos	9.67913
$s-h =$	62 5	l. sin	9.94627
			19.69050
		l. sin ½	9.84525

	h m s
L. ap. t. Sept. 30	5 55 34
Eq. of t.	— 10 14
L. m. t. Sept. 30	5 45 20

Long. $+ 5^h\ 30^m\ 46^s = 82°\ 41'\tfrac{1}{2}$ W.

139. Problem 44. *To find the hour-angle of a heavenly body when nearest to, or on, the prime vertical of a given place.*

Solution. If $d > L$, and with the same name, as for the body whose diurnal path is $n\ n'$ (Fig. 30), P Z n will be greatest, or nearest to 90°, when Z n is tangent to $n\ n'$, and consequently Z $n\ p = 90°$. We then have

$$\cos t = \frac{\tan p}{\cot L} = \frac{\tan L}{\tan d}. \tag{106}$$

If $d < L$, and with the same name, as for the body whose diurnal path is $m\ m'$, the body will be on the prime vertical at m, and P Z $m = 90°$; whence we have

$$\cos t = \frac{\tan\ d}{\tan\ L}. \tag{107}$$

If d and L are of different names, the diurnal circle intersects the prime vertical below the horizon, and the visible point nearest the prime vertical is in the horizon. The hour-angle of this point can be found by (104), omitting the effect of refraction,

$$\cos t = -\tan d \tan L.$$

Altitudes less than 8°, however, are to be avoided.

If $d = L$, the diurnal circle passes through the zenith, and the body would be on the meridian and prime vertical at the same instant; so that, when d and L are nearly equal, altitudes observed within a few minutes of the meridian passage of the body may be used for finding the time. It is only necessary that the change of altitude shall be sufficiently rapid.

But when the body is very near the meridian in azimuth the change of altitude is proportional, not to the intervals of time, but to the squares of the hour-angles. (Art. 150.) Hence, when the body is in such a position, the mean of several times does not correspond to the mean of the altitudes.

From the hour-angle the local time may be found by Problems 36 and 37.

Examples. (Prob. 44.)

1. Find the time of the greatest eastern elongation of Polaris, 1865, July 16, in latitudes 50° 18′ N. and 10° 27′ N. and long. 58° W.

$L =$	$+ 50° 18'$	l. tan 0.0808	$+ 10° 27'$	l. tan 9.2659
$d =$	$+ 88\ 35'.2$	l. cot 8.3922	$+ 88\ 35'.2$	l. cot 8.3922
$t =$	$- 5^h 53^m.2$	l. cos 8.4730	$- 5^h 59^m.0$	l. cos 7.6581
∗ R. A. =	1 10 .3		1 10 .3	
L. sid. t.	19 17 .1		19 11 .3	
$-S'_0$	−7 39 .9		−7 39 .9	
L. m. t.	11 37 .2		11 31 .4	

2. Find the time when the sun is on the prime vertical, 1865, June 10 A.M., in latitude 26° 15′ N., longitude 155° 16′ E. $= - 10^h\ 21^m\ 4^s$.

	$L = +26° 15'$	l. cot 0.3070	⊙'s dec. $+23° 2' 33''$	$+11''.3$
	$d = +23 \quad 0.6$	l. tan 9.6281	$-1 \quad 57$	113
	$t = -\ 2^h \ 2^m.1$	l. cos 9.9351	$23 \quad 0 \quad 36$	4
L. ap. t. June 9	21 57 .9		Eq'n of t. $-0^m \ 53^s$	$+0^s.49$
Eq'n of t.	-0 .9		-5	
L. m. t. June 9	21 57 .0 or June 10 $9^h \ 57^m.0$ A.M.		$-0 \quad 55$	

3. 1865, June 25, in lat. 40° 15′ N., long. 65° 17′ W., required the times when α Lyræ and α Aquilæ are on the prime vertical.

	α Lyræ.			*α Aquilæ.*		
	$L = +40° 15'$	l. cot 0.0723		$L = +40° 15'$	l. cot 0.0723	
	$d = +38 \quad 40 .8$	l. tan 9.9034		$d = +\ 8 \quad 31 .1$	l. tan 9.1754	
	$t = \mp\ 1^h \ 16^m.0$	l. cos 9.9757		$t = \mp\ 5^h \ 19^m.2$	l. cos 9.2477	
	$\alpha = \ 18 \quad 32 .4$			$\alpha = \ 19 \quad 44 .2$		
L. sid. t.	17 16 .4	or $19^h \ 48^m.4$		14 25 .0	or $1^h \ 3^m.4$	
$-S^0$	-6 17 .1	-6 17 .5		-6 16 .6	-6 18 .4	
L. m. t. June 25	10 59 .3	13 30 .9	June 25	8 8 .4	18 45 .0	

CHAPTER VII.

LATITUDE.

140. Problem 45. *To find the latitude from an observed altitude of a heavenly body on the meridian.*

Solution. Let the diagram (Fig. 31) be a projection of the sphere on the plane of the meridian N Z S:

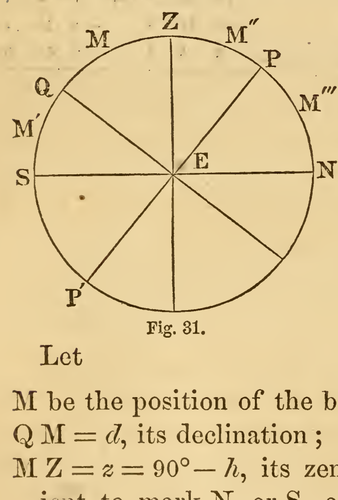

Fig. 31.

Z, the zenith;
N S, the horizon;
P, the elevated pole;
P P′, the axis of the sphere;
E Q, the equator;
Q Z, the declination of the zenith, and
N P, the altitude of the pole, are each equal to the latitude, L.

Let

M be the position of the body;
Q M $= d$, its declination;
M Z $= z = 90° - h$, its zenith distance, which it is convenient to mark N. or S., according as the zenith is north or south of the body.

From the diagram, we have Q Z = Q M + M Z

or, $$L = z + d, \tag{108}$$

which is the general formula.

If the body is at M′, *numerically*

$$L = z - d;$$

if at M″,

$$L = d - z;$$

or "the latitude is equal to the *sum* of the zenith distance and declination, when they are of the *same* name; to their *difference*, when of *different* names; and is of the same name as the greater." (Bowd., p. 166.)

If the body is at M‴, or below the pole,

$$\text{Q M}''' = 180° - d, \quad \text{and} \quad L = 180° - d - z,$$

numerically; or (108) is the correct formula, provided we use $180° - d$, or the supplement of the declination, instead of the declination.

But in this case we have also from the diagram

$$L = p + h, \tag{109}$$

as in Bowditch, p. 167.

The declination of the body must be found from the Almanac *for the time of meridian passage*. (Probs. 23, 27.) The observed altitude must be corrected for dip, refraction, &c., and the true altitude derived.

From (108) we see that an error of 1′ in the altitude will produce an error of 1′ in the resulting latitude.

Examples.

1. At sea, 1865, June 30, in lat. $2\frac{1}{2}°$ N., long. 105° 18′ W., the observed meridian altitude of the sun's lower limb was 69° 15′ 20″, sun bearing N.; index cor. +3′ 20″; height of eye 20 feet; required the latitude.

Long. $+7^h\ 1^m\ 12^s = 7^h.02$

$\underline{\odot}$ 69° 15′ 20″	In. cor. + 3′ 20″		
	S. diam. +15 46	$\odot$'s dec. 23° 10′ 21″ N.	−9″.29
+14 24	Dip − 4 24	−1 5	65.
	Ref. & p. − 18	d = 23 9 16 N.	
h = 69 29 44		z = 20 30 16 S.	
		L = 2 39 0 N.	

2. At sea, 1865, June 30, in lat. $43\frac{1}{2}°$ N., long. 150° 15′ E. $\underline{\odot}$ = 69° 15′ 20″; on meridian bearing S.; index cor. +3′ 20″; height of eye 20 feet; required the latitude.

Long. $-10^h\ 1^m\ 0^s = -10^h.02$

$\underline{\odot}$ 69° 15′ 20″	In. cor.	+ 3′ 20″	⊙'s dec. 23° 10′ 21″ N.	−9″.29
	S. diam.	+15 46	+1 33	
+14 24	Dip	− 4 24	d = 23 11 55 N.	
	Ref. & p.	− 18	z = 20 30 16 N.	
h = 69 29 44			L = 43 42 11 N.	

3. At sea, 1865, Aug. 13, 5 A.M., in lat. 25° S., long. 85° 15′ W., obs'd mer. alt. of ☽'s U. limb, 50° 18′; moon north; index cor. −2′; height of eye 16 feet; required the latitude.

Long.	+ $5^h\ 41^m\ 0^s = 5^h.68$		☽'s S. diam. 16′ 7″ +12″	
☽'s mer. pass.	Aug. 12 17 11.4	$+2^m.28$	☽'s H. par. 59 4	
Red. for long.	+13.0	11.40 1.37 18		
L. m. t.	Aug. 12 17 24.4			
G. m. t.	Aug. 12 23 5.4		☽ 50° 18′	In. cor. − 2′.0
☽'s dec.	14° 29′ 35″ N.	+7″.44	−22 .3	S. diam. −16 .3
				Dip − 4 .0
	+40	37. 3.	h'=49 55 .7	
	d = 14 30.3 N.		+37 .2	Par. and ref.
	z = 39 27.1 S.		h=50 32 .9	
	L = 24 56.8 S.			

4. At sea, 1865, Oct. 9, 5 P.M., in lat. $65\frac{1}{2}°$ N., long. 150° E.; obs'd mer. alt. of α Lyræ 63° 17′, bearing S.; index cor. +3′ 30″; height of eye 17 feet; required the latitude.

✱'s alt. 63° 17′	In. cor.	+3′.5
−1	Dip	−4 .0
h = 63 16	Ref.	− .5
z = 26 44 N.		
d = 38 40 N.		
L = 65 24 N.		

If the star bore N., the latitude would be 11° 56′ N.

5. At sea, 1865, June 18, in lat. $23\frac{1}{2}°$ N., long. 163° 0′ E.; obs'd mer. alt. of ⊙'s N. limb from N. point of the horizon, 89° 50′; index cor. +1′ 20″; height of eye 21 feet; required the latitude.

$$\text{Long. } -\underline{10^h\ 52^m\ 0^s} = -10^h.87 \qquad \odot\text{'s dec. } 23°\ 25'\ 27''\text{ N. } +\underline{3''.0}$$

$$\begin{array}{ll} \underline{\odot} \quad 89°\ 50' & \left\{\begin{array}{ll}\text{In cor.} & +\ 1'.3\\ \text{S. diam.} & +15\ .8\\ \text{Dip} & -\ 4\ .5\end{array}\right. \\ \quad\quad +12.6 & \\ h = \underline{90\quad 2.6} & \end{array} \qquad \begin{array}{rl} & -33 \\ d = & 23\quad 24.9\quad \text{N.} \\ z = & 0\quad 2.6\quad \text{N.} \\ L = & \underline{23\quad 27.5}\quad \text{N.} \end{array}$$

In this example, the true altitude of the ⊙'s centre is more than 90°; this changes the sign of z.

6. At sea, 1865, May 18, in long. 180° 0′ E., the true mer. alt. of the sun was 75° 18′; sun bearing S.; required the latitude.

$$\text{Long. } -\underline{12^h\ 0'\ 0''} \qquad \begin{array}{rl} d = & 19°\ 30'\ \text{N.} \\ z = & 14\quad 42\ \text{N.} \\ L = & \underline{34\quad 12}\ \text{N.} \end{array}$$

7. At sea, 1865, May 17, in long. 180° 0′ W.; the true mer. alt. of the sun was 75° 18′; sun bearing S.; required the latitude.

$$\text{Long. } +\underline{12^h\ 0^m\ 0^s} \qquad \begin{array}{rl} d = & 19°\ 30'\ \text{N.} \\ z = & 14\quad 42\ \text{N.} \\ L = & \underline{34\quad 12}\ \text{N.} \end{array}$$

Examples 6 and 7 are identical, the Greenwich apparent time being May 17 12^h for both. They illustrate the necessity as well as propriety of the rule for navigators near the meridian of 180°, to add 1^d to the date, when they pass from west longitude to east; to subtract 1^d from the date, when they pass from east longitude to west. For instance, May 18 5^h in long. 180° 15′ E., is identical with May 17 5^h in long. 179° 45′ W.

141. The common mode at sea of measuring a meridian altitude of the sun, is to commence observing the altitude 20 or 30 minutes before noon, repeating the operation until

the highest altitude is attained; soon after which the sun, as seen through the sight-tube of the instrument, begins to *dip*, or descend below the line of the horizon.

It is preferable, however, to find, from A.M. observations for time and by allowing for the run of the ship in the interval, the time of apparent noon by a watch, and observing the altitude at that time within 1^m or 2^m.

A meridian altitude of the moon, or a star, can be much more conveniently observed by finding beforehand the watch time of its culmination, and measuring the altitude at or very near that time.

When the sea is heavy, it is recommended to observe three or four altitudes in quick succession, within 2^m of the time of culmination.

142. If the body is changing its declination, or the observer his latitude, the *maximum* altitude is not at the instant of meridian passage; but *after*, if the body and zenith are approaching; *before*, if they are separating. Let

t be the hour-angle of this culminating point, in *minutes;*

Δd, the combined change* of declination and latitude in 1^m, if it is expressed in *seconds;* or in 1^h, if it is expressed in *minutes;*

$\Delta_0 h$, the change of altitude in 1^m from the meridian passage due solely to the diurnal rotation, (from Tab. XXXII.);

Δh, the reduction of the maximum altitude; both expressed in *seconds*.

Now in the time t

$t \Delta d$ will be the excess of altitude produced by the change of declination and latitude;

$t^2 \Delta_0 h$ (as will be shown in Art. 150), the diminution of altitude due to diurnal rotation;

---

* Their sum, if they both tend to elevate or both to depress; otherwise their difference.

and we shall have

$$\Delta h = t \Delta d - t^2 \Delta_0 h.$$

But at a point whose hour-angle is $2t$, the altitude will be the same as the meridian altitude, or

$$0 = 2t \Delta d - (2t)^2 \Delta_0 h;$$

$$\left.\begin{array}{ll} \text{whence} & t = \dfrac{\Delta d}{2 \Delta_0 h}, \\ \text{and} & \Delta h = \tfrac{1}{2} t \Delta d = \dfrac{(\Delta d)^2}{4 \Delta_0 h}, \end{array}\right\} \qquad (110)$$

which accord with the rule in Bowditch, p. 169.

Example. A ship in lat. 62° N., on March 21, sails south 14 miles per hour.

$$\Delta d = 14' + 1' = 15' \text{ per hour, or } 15'' \text{ per minute}$$

$$\Delta_0 h = 1''.0;$$

$$t = \frac{15^{m}}{2} = 7\tfrac{1}{2}^{m};$$

$$\Delta h = \frac{15}{4} \times 15'' = 56''.$$

The uncertainty of altitudes at sea makes such a correction of little practical importance; but it is generally neglected by those navigators who work out their latitudes to seconds, supposing that they have attained that degree of accuracy. In the above example, the maximum altitude of the sun would have been greater than the meridian altitude, and the latitude obtained from it in error, by nearly 1′. The sun would not have sensibly *dipped* until 9 or 10 minutes after noon.

143. A difficulty occurs at sea in measuring the meridian altitude of the sun when it passes near the zenith, on account of its very rapid change of azimuth; the change being made from east to west, 180°, in a very few minutes.

What is wanted is the angular distance of the sun from the N. or S. points of the horizon. One of these points may be sufficiently fixed by means of the compass, and then the angular distance from this point observed within 1^{m} or 2^{m} of

the meridian passage as determined by a watch regulated to apparent time.

144. From (108) we have

$$z = L - d, \tag{111}$$

by which the zenith distance may be found when the latitude and declination are given.

Also $d = L - z$, which may be used at sea for estimating the declination of a bright star from its estimated meridian altitude. If the time when it is near the meridian be also noted, and converted into sidereal time, we have the right ascension and declination of the star sufficiently near for determining what star it is.

Example.

July 16, $8^h\ 45^m$, in lat. 11° N., a bright star is seen near the meridian S., at an estimated altitude of 55°.

L. m. t. July 16	$8^h\ 45^m$	$L = 11°$	N.
S_0	7 37	$z = 35$	N.
L. sid. t.	16 22	$d = 24$	S.

The R. A. of α Scorpii (*Antares*) is $16^h\ 21^m$, and its declination 26° 7′ S.

145. Problem 46. *To find the latitude from an altitude of a heavenly body observed at any time, the local time of the observation and the longitude of the place being given.*

1*st Solution.* Reduce the observed altitude to the true altitude, and from the local time and longitude find the declination and hour-angle of the body. (Probs. 21, 34, 35.) Then in the triangle P Z M (Fig. 32) there are given

$$\text{Z P M} = t,$$
$$\text{P M} = 90° - d,$$
$$\text{Z M} = 90° - h,$$

to find

$$\text{P Z} = 90° - L.$$

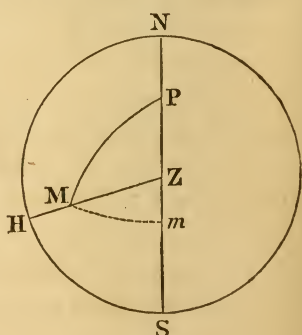

Fig. 32.

By Sph. Trig. (146), if in the triangle A B C (Fig. 33) are given a, b, and A, we find c by the formulas

$$\left.\begin{aligned} \tan \phi &= \tan b \cos A, \\ \cos \phi' &= \frac{\cos \phi \cos a}{\cos b}, \\ c &= \phi \pm \phi'; \end{aligned}\right\}$$

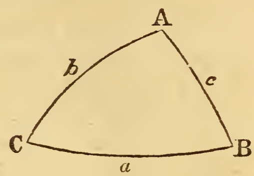

Fig. 33.

which, applied to the triangle P Z M, give

$$\left.\begin{aligned} \tan \phi &= \cot d \cos t, \\ \cos \phi' &= \frac{\cos \phi \sin h}{\sin d}, \\ 90^\circ - L &= \phi \pm \phi'. \end{aligned}\right\} \qquad (112)$$

These may be changed into a more convenient form for practice, if we put $\phi = 90^\circ - \phi''$; then

$$\left.\begin{aligned} \tan \phi'' &= \tan d \sec t, \\ \cos \phi' &= \frac{\sin \phi'' \sin h}{\sin d}, \\ L &= \phi'' \mp \phi'. \end{aligned}\right\} \qquad (113)$$

Here, observing that + and − may be rendered by N. and S. respectively, we mark ϕ'' N. or S. like the declination, and ϕ' either N. or S.; then the *sum* of ϕ'' and ϕ' when of the *same* name, their *difference* when of *different* names, is the latitude, of the same name as the greater. There are two values of L corresponding to the same altitude and hour-angle, but which, *unless* ϕ' *is very small*, will differ largely from each other; so that we may take that value which agrees best with the supposed latitude (at sea the latitude by account). When $t > 6^h$, $\phi'' > 90^\circ$, as in (97).

146. In Fig. 32, if M m be drawn perpendicular to the meridian, we shall have

$\phi = \mathrm{P}\, m,$	the polar distance of m,
$\phi'' = 90^\circ - \mathrm{P}\, m,$	the declination "
$\phi' = \mathrm{Z}\, m,$	the zenith distance "

When ϕ' is very small, (that is, when M m nearly coincides with M Z), ϕ' cannot be found with precision from its cosine. If not greater than 12°, it can be found only to the nearest minute with 5-place tables; if only 2°, it can be found only within 3'. The more nearly, then, that M m coincides with Z m, or, in other words, the nearer the body is to the prime vertical, the less accurate is the determination of the latitude. If the body is on the prime vertical, $\cos \phi' = 1$, and ϕ' cannot be found within 30'.

147. To find the effect of an error in the altitude, let

$\Delta h =$ a small change of altitude;

$\Delta \phi' =$ the corresponding change of ϕ'; it will also be numerically the change of latitude, as ϕ'' does not depend on h;

Then from the 2d of (113)

$$\cos (\phi' + \Delta \phi') = \frac{\sin \phi''}{\sin d} \sin (h + \Delta h);$$

or, since Δh and generally $\Delta \phi'$ are so small that we may take

$$\cos \Delta h = 1, \qquad \sin \Delta h = \Delta h \sin 1'',$$
$$\cos \Delta \phi' = 1, \qquad \sin \Delta \phi' = \Delta \phi' \sin 1'',$$
$$\cos \phi' - \Delta \phi'. \sin \phi' \sin 1'' = \frac{\sin \phi''}{\sin d} (\sin h + \Delta h. \cos h \sin 1'').$$

Subtracting this from the second of (113), and reducing, we have

$$\Delta \phi' = -\frac{\sin \phi'' \cos h}{\sin d \sin \phi''} \Delta h;$$

or, since

$$\frac{\cos \phi'}{\sin h} = \frac{\sin \phi''}{\sin d},$$

$$\Delta \phi' = -\Delta h. \cot \phi' \cot h. \qquad (114)$$

But in the triangle M Z m,

$$\cos \text{M Z } m = -\cos \text{M Z P} = \frac{\tan m \text{ Z}}{\tan \text{M Z}};$$

that is, Z being the azimuth,

$$-\cos Z = \frac{\tan \phi'}{\cot h}, \quad \text{or} \quad \sec Z = -\cot \phi' \cot h,$$

and therefore

$$\varDelta \phi' = \varDelta h. \sec Z. \tag{115}$$

If the body is on the meridian, $Z = 0$ or $180°$, and numerically $\varDelta \phi' = \varDelta h$.

The nearer Z is to $90°$, the greater is $\varDelta \phi'$. If $Z = 90°$, or the body is on the prime vertical, $\sec Z = \infty$, and $\varDelta \phi'$ is incalculable. If Z is near $90°$, (115) is inaccurate; since $\varDelta \phi'$ becomes too large for the assumptions

$$\cos \varDelta \phi' = 1, \qquad \sin \varDelta \phi' = \varDelta \phi'. \sin 1'';$$

so, also, in (114) if ϕ' is very small, $\varDelta \phi'$ may become large.

A star which transits the meridian near the zenith, changes its azimuth very rapidly. Unless observed on the meridian, it cannot be depended on for latitude.

148. To find the effect of an error in the time, and consequently in the hour-angle, we may take the formula in Prob. 39.

$$\sin h = \sin L \sin d + \cos L \cos d \cos t, \tag{116}$$

and letting

$\varDelta t =$ a small increase of the hour-angle, expressed in time, and

$\varDelta L =$ the corresponding change of the latitude, by a similar but more complicated process,* we shall obtain

$$\varDelta L = -\frac{15 \cos L \cos \phi'' \tan t}{\sin \phi'} \varDelta t. \tag{117}$$

But, from (97), changing the notation and regarding Z as positive toward the right,

$$\tan Z = \frac{\cos \phi'' \tan t}{\sin \varphi'};$$

and by substitution in (116),

* Such formulas can be more simply obtained by the process of differentiation.

$$\Delta L = -15\,\Delta t. \cos L \tan Z, \tag{118}$$

which requires that the azimuth should be known.

At sea the chief uncertainty of this problem is in the time, either from its imperfect determination by observation, or from unavoidable errors in allowing for the run of the ship in the interval between the observations for time and for latitude.

By (118) it appears that the effect of an error in the time is 0 when $Z = 0$ or $180°$, that is, when the body is on the meridian; and the effect is incalculable, when $Z = 90°$ or $270°$, or the body is on the prime vertical.

Moreover the effect is opposite on different sides of the meridian, and would be eliminated by two observations of the same body, or of different bodies, at the same azimuth E. and W. of the meridian.

149. 2*d Solution.* If the latitude is already approximately known, we have, (116)

$$\sin h = \sin L \sin d + \cos L \cos d \cos t;$$

whence

$$\cos (L - d = \sin h + 2 \cos L \cos d \sin^2 \tfrac{1}{2} t;$$

or since $(L - d)$ is the meridian zenith distance of the body (108), denoting it by z_0, and the meridian altitude by h_0, we have

$$\left.\begin{array}{l} \cos z_0 = \sin h_0 = \sin h + 2 \cos L \cos d \sin^2 \tfrac{1}{2} t, \\ \text{or} \\ \cos z_0 = \sin h_0 = \sin h + \cos L \cos d \operatorname{versin} t; \end{array}\right\} \tag{118}$$

in which we may use the approximate value of L in computing the term $\cos L \cos d$ versin t; which term is smaller the nearer the observation is taken to the meridian. Having found the meridian zenith distance, we may find the latitudes as in Prob. 45. If the computed value of L differs largely from the assumed value, the computation should be repeated, using this new value.

This is the method in Bowditch, page 200.

150. 3*d Solution.* When the observation is taken very near the meridian, we may find the correction to be applied to the observed altitude to reduce it to the meridian altitude, thus:

From (118) we have

$$\sin h_0 - \sin h = 2 \cos L \cos d \sin^2 \tfrac{1}{2} t,$$

whence, by Sph. Trig. (106),

$$\cos \tfrac{1}{2} (h_0 + h) \sin \tfrac{1}{2} (h_0 - h) = \cos L \cos d \sin^2 \tfrac{1}{2} t.$$

But h_0 and h differing very little, we may put

$$\cos \tfrac{1}{2} (h_0 + h) = \cos h_0 = \sin z_0 = \sin (L - d),$$

so that

$$\sin \tfrac{1}{2} (h_0 - h) = \frac{\cos L \cos d \sin^2 \tfrac{1}{2} t}{\sin (L - d)}. \tag{119}$$

Put $\varDelta h = h_0 - h$, the reduction of the observed to the meridian altitude, or, as it is usually called, "the reduction to the meridian;" and, since $\varDelta h$ and t are quite small, put

$\sin \frac{1}{2} \varDelta h = \frac{1}{2} \varDelta h. \sin 1''$, ($\varDelta h$ being expressed in seconds of arc),
$\sin \frac{1}{2} t = \frac{1}{2} t \times 15 \sin 1''$, ($t$ " " " of time),

then (119) reduces to

$$\varDelta h = \frac{112.5 \sin 1'' \cos L \cos d}{\sin (L - d)} \times t^2;$$

or, since $\sin 1'' = 0.000004848$,

$$\varDelta h = \frac{0''.000545 \cos L \cos d}{(\sin L - d)} \times t^2 \quad (t \text{ in seconds}).$$

In this formula t is in seconds of time; but if, as is usual, t is expressed in *minutes*, we must put $(60\, t)^2$ for t^2, so that we have

$$\varDelta h = \frac{1''.96349 \cos L \cos d}{\sin (L - d)} \times t^2 \tag{120}$$

If $t = 1^m$, the formula expresses the change of altitude in one minute from the meridian. Representing this by $\varDelta_0 h$, we have

$$\varDelta_0\, h = \frac{1''.96349 \cos L \cos d}{\sin (L - d)} \tag{121}$$

and

$$\left.\begin{aligned} \varDelta\ h &= t^2.\varDelta_0\, h, \\ h_0 &= h + \varDelta\ h, \text{ the meridian altitude.} \end{aligned}\right\} \tag{122}$$

Whence the latitude is found as by a meridian altitude (Prob. 45).

Bowditch's Tab. XXXII. contains the values of $\varDelta_0\, h$ for each 1° of declination from 0 to 24°, and each 1° of latitude from 0 to 70°; except when $L-d < 4°$, for then $\varDelta\ h$ is so large that (120) and (121) become inaccurate. In this case the body is near the zenith, and altitudes out of the meridian do not afford a reliable determination of the latitude.

Bowditch's Table XXXIII. contains t^2 for each 1^s of t from 0 to 13^m.

When h is small, the reduction to the meridian may be found by this method quite accurately even when t is as great as 12^m. If h is near 90°, t must be taken within much narrower limits. (Bowd., p. 202.) Indeed, in this case z_0, or its equal $(L-d)$, is very small, and consequently $\varDelta_0\, h$ becomes large. Unless then t is sufficiently small, $\varDelta\ h$ will be too great for the assumption $\sin \frac{1}{2}\ \varDelta\ h = \varDelta\ h \sin 1''$.

If $d > L$, $\sin (L-d) = \sin z_0$ is negative; that is, z_0 will have a different name or sign from L (Art. 140). Properly h, h_0, and $\varDelta_0\, h$ would also become negative to correspond. Still, however, we shall have numerically

$$h_0 = h + \varDelta\ h.$$

We may therefore disregard the sign of $L-d$ in (121) and consider h and h_0 as always positive.

If the star is observed at its lower culmination, then t will be the hour-angle from the lower branch of the meridian, and for d we may use $180° - d$ (Art. 140). $\varDelta_0\, h$ and $\varDelta\ h$ are then numerically subtractive.

Examples. (Prob. 46.)

1. At sea, 1865, July 17 1^h P. M., in lat. 36° 38′ S., long. 105° 18′ E., by account; time by Chro., $5^h\ 47^m\ 14^s$; ☉, 30° 15′; N. W'y; index cor. + 2′ 30″; height of eye, 17 feet; Chro. cor. (G. m. t.) + $14^m\ 3^s$; required the latitude.

By (113)

T. by Chro. + 12^h,	$17^h\ 47^m\ 14^s$		☉'s *dec.*		*Eq'n of t.*
Chro. cor.	+ 14 3		+21° 20′ 29″	−25″.19	$-5^m\ 43^s.8$ $-0^s.230$
G. m. t. July 16	18 1 17 = 18.021		−7 34	252, 202	−4.1 (2.3, 1.8)
−Long.	+ 7 1 12		+21 12 55		−5 47.9
L. m. t. July 17	1 2 29	☉	30° 15′	In. cor. + 2′.5; S. diam + 15′.8	dip. −4′.0; ref.&par. −1′.5
Eq. of t.	− 5 48		+ 13		
L. ap. t.	0 56 41		h = 30 28	l. sin	9.70504
t^* =	14° 10′ 15″	l. sec	0.01342		
d =	21 12.9 N.	l. tan	9.58903	l. cosec	9.44145
ϕ'' =	21 49.1 N.	l. tan	9.60245	l. sin	9.57016
ϕ' =	58 37.0 S.			l. cos	9.71665
L =	36 48 S.				

If we suppose an uncertainty of 3′ in the altitude and 20′ in the longitude, by (115) and (118)

	l. cot $(-h)$	0.2304 n		l. cos L	9.903
	l. cot ϕ'	9.7853	$-\Delta t = -20'$	log	1.301 n
Z = S. 164° 40′ W.	l. sec Z	0.0157 n		l. tan Z	9.438 n
$\Delta h = +3'$	log	0.477	$\Delta L = +4'.4$	log	0.642
$\Delta L = -3'.1$	log	0.493 n			

That is, an increase of 3′ in the altitude will numerically decrease the latitude 3′.1; and a numerical increase of 20′ in the assumed longitude will increase the latitude 4′.4. This may be conveniently expressed in the following way:

* Instead of changing t into arc, we may enter col. P. M. of Tab. XXVII. with $2t = 1^h\ 53^m\ 22^s$.

Long. $105° 18' \pm 20'$ E.; $\underline{\odot}$, $30° 15' \pm 3'$
$L = 36° 48' \pm 4'.4 \mp 3'.1$ S.

By (119)

		log 2 0.30103	(or l. versin 8.48330)
$t = 0^h 56^m 41^s$		2 l. sin ½ 8.18227	
$d = 21° 12'.9$	N.	l. cos 9.96953	
		8.45283	
1st $L = 36$ 38	S.	l. cos 9.90443	
		log 8.35726	.02276
$h = 30$ 28			sin .50704
$Z_0 = 58$ 0.5	S.		cos .52980
2d $L = 36$ 47.6	S.	l. cos 9.90345	
		log 8.35628	0.2271
$Z_0 = 58$ 0.7	S.		cos .52975
3d $L = 36$ 47.8	S.		

2. At sea, 1865, Jan. 5, 6^h P. M., in lat. 50° 36′ N., long. 135° 25′ W. (by account), time by Chro. $3^h 10^m 15^s$; Chro. cor. (G. m. t.) $-18^m 56^s$; Obs'd alt. of Mars, 45° 18′; S. E'y; index cor. $-3'$; height of eye, 19 feet; required the latitude. (113)

	h m s			h m s	s
T. by Chro. $+12^h$,	15 10 15				
Chro. cor.	—18 56		Mars' R. A.	3 55 25.1	+ 0.037
G. m. t. Jan. 5	14 51 19	$= 14^h.855$		+ 0.5	.37
S_0	19 0 22.1			3 55 25.6	.14
					3
Red. for G. m. t.	+ 2 26.5		Mars' dec.	+ 23° 0′ 30″	+ 0″.50
G. sid. t.	9 54 7.6			+ 7	
—Long.	—9 1 40			+ 23 0 37	
L. sid. t	0 52 27.6	$h' = 45° 18'$	In. cor. — 3′		
Mars' R. A.	3 55 25.6	— 8	dip. & ref. — 5		
	$t = -$ 3 2 58	$h = 45$ 10		l. sin	9.85074

or 45° 44′ 30″	l. sec 0.15621	
$d =$ 23 0 37 N.	l. tan 9.62807	l. cosec 0.40793
$\phi'' =$ 31 19.3 N.	l. tan 9.78428	l. sin 9.71588
$\phi' =$ 19 25.5 N.		l. cos 9.97455
$L =$ 50 45 N.		

If $\Delta h = +5'$ and $\Delta \lambda = +15'$, $\Delta t = -15'$; and by (115) and (118)

	l. cot $(-h)$ 9.9975 n		l. cos L 9.801
	l. cot ϕ' 0.4527		$-\Delta t = +15'$, log 1.176
$Z =$ N. 110° 46′ E.	l. sec Z 0.4502 n		l. tan Z 0.421 n
$\Delta h = +$ 5′	log 0.699		$\Delta L = -25'.0$, log 1.398 n
$\Delta L = -14'.1$	log 1.149 n		

3. 1865, Feb. 17, near noon, at the light-house, W. end of St. George's Island, Apalachicola Bay, long. 85° 5′ 15″ W.; 5 observations with sextant No. 1, art. hor'n No. 3, A end toward observer:

T. by Chro. $0^h\ 16^m\ 21^s.6$; 2⊙ 96° 14′ 44″, (S.); in. cor. +2′ 30″; Chro. cor. (L. m. t.) $-18^m\ \overline{30}^s.4$; Bar. 30.48, Ther. 43°.

By (113)

T. by chro.	$0^h\ 16^m\ 21^s.6$	⊙'s *dec.*		*Eq. of t.*	
Chro. cor.	−18 30 .4	−11° 51′ 52″.8	+52″.71	−14′ 15″.22	$+0^s.204$
L. m. t. Feb. 16	23 57 51 .2	+4 57 .1	263 .55		1 .020
Long.	+5 40 21	−11 46 55 .7	31 .63	+1 .15	.122
			1 .58		6
G. m. t. Feb. 17	5 38 12	$=5^h.637$	.37	−14 14 .07	
Eq. of t.	−14 14 .1	⊙ 48° 7′ 22″	½ In. cor.	+ 1′ 15″	
L. ap. t.	23 43 37 .1		S. diam.	+16 30	
	or	+16 57	Ref.	− 54	
	−0 16 22 .9		Par.	+ 6	
		$h=$ 48 24 19			

$t =$ 4° 5′ 43″.5	l. sec 0.0011104	l. sin h 9.8738198
$d = -11$ 46 55 .7	l. tan 9.3192842 n	l. cosec 0.6899636 n
$\phi'' = -11$ 48 41 .2	l. tan 9.3203946 n	l. sin 9.3111004 n
$\phi' = +41$ 26 10 .2		l. cos 9.8748838
$L = +29$ 37 29		

By (121) and (122)

$1^s.96349$	log 0.2930	
$L = +29°$ 37	l. cos 9.9392	$h =$ 48° 24′ 19″
$d = -11$ 47	l. cos 9.9908	$\Delta_0 h = +$ 11 18
$L - d = +41$ 24	l. cosec 0.1796	$h_0 =$ 48 35 37
$\Delta_0 h =$ 2″.527	log 0.4026	$Z_0 = +41$ 24 23
$t = -16^m.382$	2 log 2.4287	$d = -11$ 46 56
$\Delta h = +678''.1$	log 2.8313	$L = +29$ 37 27

LATITUDE BY CIRCUM-MERIDIAN ALTITUDES.

151. PROBLEM 47. *To find the latitude from a number of altitudes observed very near the meridian, the local times being known.*

Solution. By (122) we see that very near the meridian the altitude of a body varies very nearly as the square of its hour-angle. Hence we cannot regard the mean of several altitudes as corresponding to the mean of the times, since this is assuming that the altitude varies as the hour-angle, Let

h_1, h_2, h_3, &c., be the several altitudes;

t_1, t_2, t_3, &c., the corresponding hour-angles expressed in *minutes;*

and we have as the reduction of each altitude to the meridian, and the deduced meridian altitude,

$$\left.\begin{array}{ll} \Delta_1 h = t_1^2 . \Delta_0 h & h_0 = h_1 + \Delta_1 h \\ \Delta_2 h = t_2^2 . \Delta_0 h & h_0 = h_2 + \Delta_2 h \\ \Delta_3 h = t_3^2 . \Delta_0 h \text{ \&c.} & h_0 = h_3 + \Delta_3 h \end{array}\right\} \text{\&c.} \qquad (123)$$

Thus the meridian altitude may be derived from each altitude, and the mean of all these meridian altitudes taken as the correct meridian altitude. But the following is a more expeditious method :—

If n is the number of observations, the mean value of h_0 will be

$$h_0 = \frac{h_1 + h_2 + h_3 + \ldots h_n}{n} + \frac{\Delta_1 h + \Delta_2 h + \Delta_3 h + \ldots \Delta_n h}{n}$$

or,

$$h_0 = \frac{h_1 + h_2 + h_3 + \ldots h_n}{n} + \frac{t_1^2 + t_2^2 + t_3^2 + \ldots t_n^2}{n} \Delta_0 h \qquad (124)$$

Whence the rule :—

Take the mean of the squares of the hour-angles in *minutes* (Tab. XXXIII., Bowd.); multiply it by the change of altitude in 1^m from the meridian (Tab. XXXII.); and add the

product to the mean of the altitudes. The result is the mean meridian altitude required. (Bowd., p. 201.) From the meridian altitude thus found deduce the latitude as from any other meridian altitude. (Prob. 45.) Strictly, however, the declination to be used is that which corresponds to the mean of the times, and the hour-angles, t, are intervals of *apparent* time for the sun, and of *sidereal* time for a fixed star.

152. It is unnecessary to reduce each observed altitude separately to a true altitude; as the reductions, excepting slight changes of refraction and parallax, are the same for all, and may be computed for the mean of the observed altitudes, and applied to this mean with the reduction to the meridian.

153. Should it be desirable to compare the several observations with each other, and test their agreement, it will be sufficient to compute the several reductions to the meridian, $\Delta_1 h$, $\Delta_2 h$, $\Delta_3 h$, &c., and apply them separately to the *readings* of the instrument; or to the *half-readings* when the altitudes are observed with an artificial horizon: applying, also, the semidiameter when both limbs of the body are observed.

154. If the altitudes are taken on both sides of the meridian, and at nearly corresponding intervals, a small error in the local time will but slightly affect the result; for such error will make the estimated hour-angles and the corresponding reductions on one side of the meridian too large, and on the other side too small. (Bowd., p. 203.)

155. This method is rarely used at sea, as a single altitude on or near the meridian suffices. No increase of the number of observations will diminish at all the error of the dip, which affects alike each observation and the mean of all.* But on

* Such an error is called *constant;* those which affect the several observations differently are called *variable.*

land it is preferable to measure a number of altitudes at the same culmination of the body, and thus diminish the "error of observation." Altitudes of the sun are used, but the best determinations are from the altitudes of a bright star. To facilitate the operations, and avoid mistaking one star for another, it is well to compute the altitude approximately beforehand. (Art. 144.)

If an artificial horizon is employed, the error of the roof is partially eliminated by making two sets of observations with the roof in reversed positions.

156. If two stars are observed which culminate at nearly the same altitude, one north, the other south of the zenith, the error of the instrument is nearly eliminated; for such error (except *accidental* error of graduation) will make the latitude from one of the stars too great, and that from the other too small by very nearly the same amount; the more nearly, the less the difference of the altitudes. The error peculiar to the observer is also eliminated.

If the observations are made with an artificial horizon, the error of the roof is eliminated, if the *same* end is toward the observer in both sets of observations.

157. Bowditch's Tab. XXXII. extends only to $d = 24°$ If a star is used whose declination is beyond this limit, or if greater precision than the table affords is required, $\Delta_0 h$ may be computed for the star and place by (121).

$$\Delta_0 h = \frac{1''.9635 \cos L \cos d}{\sin (L - d)}$$

158. If the observations are made at the lower culmination of the star, we have only to use in the formulas $180° - d$ instead of d. (Art. 140.)

159. The altitudes observed at the same culmination are very nearly the same. To render the measurements independent, after each observation move slightly the tangent screw of the instrument. With the sextant, it is best to

make the final motion of the tangent screw at each observation always in the same direction, for example, in advance.

EXAMPLES. (Prob. 47.)

1. 1865, Jan. 10 0^h. Circum-meridian altitudes of ⊙ observed at the Custom-House, Key West, Florida, 45″ N. of Light-House: lat. 24° 34′ N., long. 81° 48′ 37″ W.

T. by Watch. *Sextant No.* 1. *Art. hor. No.* 1.

h m s	° ′ ″	
0 5 0	2 $\overline{⊙}$ 87 41 30	*A* end of Hor.
5 30	41 30	
5 50	41 35	
7 8	41 40	*B*.
7 30	41 50	
7 55	41 55	
9 0	2 $\underline{⊙}$ 86 36 25	
9 25	36 40	
9 45	36 30	
12 2	35 20	*A*.
12 25	34 50	
12 45	34 40	

Comparisons.

	h m s	h m s
Chro. 1843	4 58 24	6 26 41
Watch	11 39 20	1 7 30
C—W	+5 19 4	5 19 11

Chro. cor. (L. m. t.) $-5^h\ 18^m\ 34^s.0$

⊙'s diam. { off arc 32′ 47″.0 / on arc 32 6 .5 }

In. cor. + 20 .2

Bar. 30.21

Ther. 78°

	h m s	° ′ ″
T. by W.	0 8 41.2	2 $\overline{⊙}$ 87 41 40
W. cor (L. ap. t.)	—7 30.1	2 $\underline{⊙}$ 86 35 44
L. ap. t. Jan. 10	0 1 11.1	2 ⊙ 87 8 42
Long.	+5 27 14.5	
G. ap. t. Jan. 10	5 28 25.6	
	5.474	

(At $0^h\ 9^m$) C—W.	$5^h\ 19^m\ 6^s.3$	
Chro. cor. (L. m. t.)	—5 18 34 .0	
W. cor. (L. m. t.)	+ 0 32 .3	
Eq'n of t.	$+7^m\ 56^s.96$	$+0^s.998$
	+5.46	{ 4 .99 / .47 }
	+8 2.42	
Watch t. of ap. 0^h	$0^h\ 7^m\ 30^s.1$	
Hourly ch.	—3 .7	
⊙'s *dec.*	—21° 54′ 43″.3	+22″.90
	+2 5 .3	{ 114 .5 / 9 .16 / 1 .60 / 9 }
	—21 52 38	

t	t^2
— $2^m\ 30^s$	6.2
2 0	4.0
1 40	2.8
0 22	0.1
0 0	0.0
+ 0 25	0.2
1 30	2.2
1 55	3.7

t	t^2		
m s		° ′ ″	
2 15	5.1	⊙ 43 34 21	½ In. cor. + 10″.1
4 32	20.6		Ref. —58 .5
4 55	24.2	— 52	Par. + 6 .3
5 15	27.6		1″.9635 log 0.2930
	96.7	$\bar{h}$ = 43 33 29	L=+24° 34′ l. cos 9.9588
Δh=	8.06 × 2″.28	= + 18	=—21 52 l. cos 9.9657
		h_0= 43 33 47	L—d=+46 27 l. cosec 0.1398
		z_0=+46 26 13	$\Delta_0 h$= 2″.28 log 0.3573
		d =—21 52 38	
Custom-House,		L =+24 33 35	Light-House, L=+24° 32′ 50″

2. 1843, January 31 (civil date). Circum-meridian altitudes of ⊙ observed at E. Base station, Mullet Key in Tampa Bay; lat. 27° 37′ N., long 5ʰ 30ᵐ 50ˢ W.

T. by Chron.	*Sextant No.* 2.		*Art. hor. No.* 1.
h m s		° ′ ″	
0 5 30	2 $\overline{⊙}$	90 20 20	A end toward obser.
6 10		21 30	
6 45		22 20	
8 42	2 $\underline{⊙}$	89 19 50	
9 12		20 10	
9 38		20 40	
11 00		21 36	B end toward obser.
11 30		22 10	
11 52		22 30	Index cor. + 1′ 5″
13 15	2 $\overline{⊙}$	90 27 50	
13 45		28 10	Chron. cor. — 1ᵐ 39ˢ.3
14 17		28 20	
15 25		28 30	Bar. 29.95
15 50		28 40	
16 18		28 40	Ther. 72°
17 53	2 $\underline{⊙}$	89 23 0	
18 20		22 50	
18 40		22 50	
20 50		21 0	A end toward obser.
21 12		20 40	

T. by Chron.	*Sextant No. 2.*	*Art. hor. No. 1.*
h m s	° ′ ″	
0 21 37	89 20 30	
22 50	2 ⊙ 90 24 10	
23 20	23 40	
23 40	23 30	

Eq. of Time	$+13^m\ 44^s.19 + 0^s.374$		$1''.9635$	log 0.2930
	$+2^s.07$	1.87	$L = +27°\ 37'$	l. cos 9.9475
	$+13^m\ 46^s.3$	.19	$d = -17°\ 25'$	l. cos 9.9796
—Chron. cor.	$+\ 1^m\ 39^s.3$	.01	$L-d = +45°\ \ 2'$	l. cosec 0.1503
Chr. T. ap. Noon	$+15^m\ 26^s.$		$\varDelta_0\ h = 2''.346$	log 0.3704

t	t^2	(h)	$2''.346\ t^2$	S. diam.	(h_0)
m s		° ′ ″	′ ″	′ ″	° ′ ″
—9 56	98.7	45 10 10	+3 51	—16 16	44 57 45
9 16	85.9	10 45	3 21		50
8 41	75.4	11 10	2 57		51
6 44	45.3	44 39 55	1 46	+16 16	57
6 14	38.9	40 5	1 31		52
5 48	33.6	40 20	1 19		55
4 26	19.7	40 45	0 46		47
3 56	15.5	44 41 5	0 36		57
3 34	12.7	41 15	30		61
2 11	4.8	45 13 55	11	—16 16	50
1 41	2.8	14 5	7		56
1 9	1.3	14 10	2		56
—0 1	0.0	14 15	0		59
+0 24	0.2	14 20	0		64
0 52	0.8	14 20	2		66
2 27	6.0	44 41 30	14	+16 16	60
2 54	8.4	41 25	20		61
3 14	10.5	41 25	25		66
5 24	29.2	40 30	1 8		54
5 46	33.3	40 20	1 18		54
6 11	38.2	40 15	1 30		61
7 24	54.8	45 12 5	2 9	—16 16	58
7 54	62.4	11 50	2 26		70
+8 14	67.8	11 45	+2 39		68

Mean	—0 32	Mean ⊙'s U. L.	44 57 58
Long.	5^h 30 50	⊙'s L. L.	57
G. ap. T.	5 30 18	A	44 57 57
		B	58

☉'s Dec.	$-17°\ 28'\ 26''.0$	$+41''.55$	$h'_0 =$	$44°\ 57'\ 57''.4$
	$+\ \ \ 3\ 48.7$	207.75	½ In. cor.	$+32.5$
$d =$	$-17\ 24\ 37.3$	20.71	Ref.	-56.2
$z_0 =$	$+45\ \ 2\ 20$	.21	Par.	$+6.2$
Lat. $=$	$+27\ 37\ 43$		$h_0 =$	$44\ 57\ 40$

3. 1865, May 22, 9^h, circum-meridian altitudes of *a* Virginis (*Spica*) at Light-House on St. George's Island, Apalachicola Bay, Florida, lat. 29° 37′ N., long. 85° 5′ 15″ W.

T. by Chro. h m s	*Sextant No.* 1. ° ′ ″	*Art. Hor. No.* 1.		
3 28 56	2 *alt.* 99 43 50	*A. end*	In. cor. —3′ 0″	
31 24	47 50		Bar. 30.04,	Ther. 73°
33 36	51 0		Chro. cor. (L.m.t.)	$+5^h\ 35^m\ 32^s.9$
34 56	53 40		Long.	+5 40 21
37 8	53 40			
38 58	55 30	*B. end*		h m s
42 45	55 30		✳'s R. A.	13 18 7.8
44 33	52 10		$-S_0$	—4 0 30.0
48 21	46 50		—Red for λ	—55.9
51 25	43 50		Sid. int. from 0^h	9 16 41.9
	99 50 23		Red.	—1 31.2
	$h' =$ 49 55 12	½ In. cor. —1′ 30″	L. m. t. of transit	9 15 10.7
	—2 17	Ref. — 47	—Chro. cor.	—5 35 32.9
	$h =$ 49 52 55		Chro. t. of transit	3 39 38

(*Mean*) t m s	(*Sid.*) t m s	t^2		
—10 42	—10 44	115.2	1″.9635	log 0.2930
8 14	8 15	68.1	$L = +$ 29° 37′	l. cos 9.9391
6 2	6 3	36.6	$d = -$ 10 28	l. cos 9.9927
4 42	4 43	22.2	$L-d = +$ 40 5	l. cosec 0.1912
2 30	2 30	6.2	$\Delta_0 h =$ 2″.606	log 0.4160
0 40	0 40	0.4	$t^2 =$ 49.86	log 1.6979
+3 7	+3 7	9.7	$\Delta h = +$ 2′ 10″	log 2.1139
4 55	4 56	24.3	$h =$ 49° 52 55	
8 43	8 44	76.3	$h_0 =$ 49 55 5	
11 47	11 49	139.6	$z_0 = +$ 40 4 55	
		49.86	$d = -$ 10 27 34	
			$L = +$ 29 37 21	

160. PROBLEM 48. *To find the latitude from an observed altitude of Polaris or the North Pole-star.*

Solution. The formulas (112) of Prob. 46,

$$\tan \phi = \cot d \cos t$$

$$\cos \phi' = \frac{\cos \phi \sin h}{\sin d}$$

$$90° - L = \phi \pm \phi'$$

can be greatly simplified in the case of the Pole-star, since its polar distance is only 1° 25′.

Putting $d = 90° - p$ and $\phi' = 90° - \phi''$, we have

$$\tan \phi = \tan p \cos t$$

or,

$$\left.\begin{aligned} \phi &= p \cos t \text{ (within } 0''.5) \\ \sin \phi'' &= \sin h \frac{\cos \phi}{\cos p} \\ L &= \phi'' - \phi, \end{aligned}\right\} \qquad (125)$$

the 2d value of L, or $(180° - \phi'' - \phi)$, being excluded, as it exceeds 90°. p and ϕ are so small, that the cosine of each is nearly 1, and consequently

$$\sin \phi'' = \sin h \quad \text{and} \quad \phi'' = h, \text{ nearly.}$$

Thus we have

$$\left.\begin{aligned} \phi &= p \cos t \\ L &= h - \phi \end{aligned}\right\} \qquad (126)$$

If t is more than 6^h or less than 18^h, $\cos t$ is negative, and we have numerically

$$L = h + \phi.$$

Let S represent the sidereal time, and a the right ascension of the star, then

$$t = S - a \quad \text{and} \quad \phi = p \cos (S - a).$$

If we consider the right ascension and polar distance of the star to be constant, ϕ may be computed and tabulated for different sidereal times (right ascensions of the meridian), as in Bowditch, p. 206, and "Tab. I. for the Pole-star" in

the British Nautical Almanac. But owing to the change of right ascension and declination, such a table requires correction for each year. It is better to take the *apparent* right ascension and declination from the Almanac, and compute t and ϕ.

ϕ may be found approximately in the traverse table (Tab. II.) in the *Lat. col.*, by entering the table with t as a *course*, and p as a *distance*.

161. Formulas (126) may be derived from Fig. 34, by regarding P M m as a plane triangle, and Z m = Z M. The first produces no error greater than 0″.5. The error of the second is evidently greater the greater the altitude, or the latitude. This error, however, will not be more than 0′.5 in latitudes less than 20°, nor more than 2′ in latitudes less than 60°.

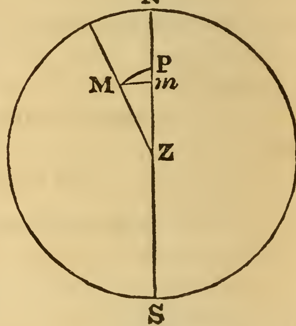

Fig. 34.

162. We may use (125) with more exactness, but these formulas may be modified so as to facilitate computation.

Put $$\phi'' = h + \Delta h$$

then, changing the 2d of (125) to a logarithmic form, we have

$$\log \sin (h + \Delta h) = \log \sin h + \log \cos \phi - \log \cos p,$$

or

$$\log \sin (h + \Delta h) - \log \sin h = \log \sec p - \log \sec \phi.$$

But Δh being very small, representing by $D_{\prime\prime}$ the change of log sin h for 1″, we have, with Δh in *seconds*,

$$\log \sin (h + \Delta h) - \log \sin h = \Delta h \times D_{\prime\prime};$$

whence, by substituting in the preceding, we obtain

$$\Delta h = \frac{\log \sec p - \log \sec \phi}{D_{\prime\prime}} = \frac{\log \cos \phi - \log \cos p}{D_{\prime\prime}}. \quad (127)$$

The difference of the log secants, or log cosines, of p and ϕ is readily taken from the table by inspection. $D_{\prime\prime}$ for log sin h is usually given in tables of 7 decimal places, and hence Δh is readily found.

We have then

$$\left.\begin{array}{l}\phi = p \cos t \\ L = h + \Delta h - \phi\end{array}\right\} \qquad (128)$$

If $D_{\prime}$ is the change of log sin h for 1′, then in *minutes*

$$\Delta h = \frac{\log \sec p - \log \sec \phi}{D_{\prime}}. \qquad (129)$$

163. The British Nautical Almanac contains three tables for the reduction of altitudes of Polaris, from which they may be found to the nearest second.

164. Altitudes of Polaris may often be observed at sea, with some degree of precision, during twilight, when the horizon is well defined, and the latitude found from them within 3′ or 4′.

Examples. (Prob. 48.)

1. At sea, 1865, March 31, $7^h\ 15^m\ 19^s$, mean time in long. 160° 15′ E.; obs'd alt. of *Polaris* 38° 18′; index cor. +3′; height of eye 17 feet: what is the latitude? (128)

	h m s					
L. m. t. March 31	7 15 19		Long.	−10 41 0 (h m s)		
S_0	0 35 29					
Red. for long.	−1 45					
Red. of L. m. t.	+1 11	$h'=$	38° 18′	In. cor. +3		
L. sid. t.	7 50 14		−2	Dip −4		
*'s R. A.	1 9 14	$h=$	38 16	Ref. −1		
	$t=$ 6 41 0	$t=$	100° 15′		l. cos 9.250n	
		$-p=-$	1 24.5		log 1.927n	l. sec .00013
		$-\phi=+$	15.0		log 1.177	l. sec 0
		$\Delta h=+$	0.8	$= \dfrac{\text{l. sec } p - \text{l. sec } \phi}{D_{\prime}} =$		16
		$h=$	38 16			
		$L=+$	38 32			

2. 1865, May 22, 9^h; altitudes of *Polaris*, at light-house on St. George's Island, Apalachicola Bay: lat. 29° 37′ N.; long. 85° 5′ 15″ W.; sextant No. 1, index cor. —3′ 0″; Art. Hor. No. 1; Bar. 30.04, Ther. 73°; Chro. cor. (L. m. t.) $+5^h\ 35^m\ 33^s$.

	T. by Chro.	2 × alt. (A. end of Hor.)		T. by Chro.	2 × alt. (B. end of Hor.)
	h m s	° ′ ″		h m s	° ′ ″
	3 17 25	56 32 20		4 2 18	56 34 0
	19 26	32 30		6 3	34 20
	21 39	32 50		8 27	34 50
	30 35	32 30		10 15	34 50
				14 40	35 40
T. by Chro.	3 22 16	$2h'$=56 32 32	T. by Chro.	4 8 21	$2h'$=56 34 44
Chro. cor.	+5 35 33	h'=28 16 16	Chro. cor.	+5 35 33	h'=28 17 22
L. m. t.	8 57 49	½ In. cor. —1 30	L. m. t.	9 43 54	½ In. cor. —1 30
S_0	4 0 30	Ref. —1 43	S_0	4 0 30	Ref. —1 43
Red. for λ	+56	h=28 13 3	Red. for λ	+56	h=28 14 9
Red. for m. t.	+1 28		Red. for m. t.	1 36	
L. sid. t.	13 0 43		L. sid. t.	13 46 56	
∗'s R. A.	1 9 32		∗'s R. A.	1 9 32	
$t=$	11 51 11		$t=$	12 37 24	
	177° 47′ 45″	l. cos 9.99968 n		189° 21′ 0″	l. cos 9.99419 n
$-p=-$	1 24 44	log 3.70621 n	$-p=-$	1 24 44	log 3.70621 n
$-\phi=+$	1 24 40	log 3.70599	$-\phi=+$	1 23 37	log 3.70040
$\Delta h=$	0	l. sec p 1319	$\Delta h=+$	1	l. sec p 1319
$h=$	28 13 3	l. sec ϕ 1317	$h=$	28 14 9	l. sec ϕ 1284
$L=+$	29 37 43	($\Delta_{''}$=39.2) 2	$L=+$	29 37 47	($\Delta_{''}$=39.2) 35

l. sec p and l. sec ϕ are expressed in units of the 7th place of decimals.

CHAPTER VIII.

THE CHRONOMETER.—LONGITUDE.

165. Astronomically the longitude of a place is the difference of the local and Greenwich times of the same instant. It is *west* or *east*, according as the Greenwich time is greater or less than the local time. (Art. 73.)

The *mean solar*, the *apparent*, or the *sidereal* times of the two places may be thus compared.

166. A *chronometer* is simply a correct time-measurer, but the name is technically applied to instruments adapted to use on board ship. It is here used more generally, as including clocks which are compensated for changes of temperature.

A *mean time* chronometer is one regulated to mean time; that is, so as to gain or lose daily but a few seconds on mean time.

A *sidereal* chronometer is one regulated to sidereal time.

167. A chronometer is said to be regulated to the local time of any place, when it is known how much it is too fast, or too slow, of that local time, and how much it gains or loses *daily*. The first is the *error* (on local time); the second is the *daily rate*. Both are + if the chronometer is *fast* and *gaining*.

It is preferable, however, to use the *correction* of the chronometer, which is the quantity to be applied to the chronometer time to reduce it to the true time, and its *daily change*. Both are + when the chronometer is *slow* and *losing*.

They will be designated by c and Δc.

A chronometer is said to be regulated to Greenwich time, when its correction on Greenwich time and its daily change are known.

If c_0 is the chro. cor. to reduce to Greenwich time, and c, the chro. cor. to reduce to the time of a place whose longitude is λ (+ if west).

$$c_0 = c + \lambda, \quad \text{or} \quad c = c_0 - \lambda; \tag{130}$$

so that the one can readily be converted into the other.

168. If the correction of the chronometer at a given date, and its daily change, are known, the correction at another date can easily be found. For let

c be the given correction at the date T,
c', the required correction at the date T',
$t = T' - T$, expressed in days,
Δc, the daily change;

then
$$c' = c + t.\,\Delta c. \tag{131}$$

t is negative if the date for which the correction is required is before that for which it is given.

If Δc is large, t must include the parts of a day in the elapsed time.

Δc may be given for two different dates, and vary in value. It may then be interpolated for the middle date between the two of this problem.

Thus, if $\Delta' c$ be a second value determined n days after the first, the daily variation of Δc, regarded as uniform, will be

$$\frac{\Delta' c - \Delta c}{n}. \tag{132}$$

Representing this by $\Delta_2 c$, we have for the mean daily change of the chronometer correction during the period t, or that at the middle date,

$$\Delta c + \tfrac{1}{2}\, t.\,\Delta_2 c,$$

and the required chronometer correction,

$$c' = c + t.\, \varDelta c + \tfrac{1}{2} t^2.\, \varDelta_2 c. \qquad (133)$$

When the chronometer is in daily use, it is convenient to form a table of its correction for each day at a particular hour. For a stationary chronometer, the most convenient hour is 0^h of local time; for a Greenwich chronometer, 0^h of Greenwich time.

Examples.

1. Chro. 1675, regulated to Greenwich mean time; 1865, Jan. 15, 0_h; correction $+1^h\ 1_m\ 25^s.0$; daily change $-7^s.65$; required the correction, Jan. 26, 6^h.

Jan. 15, 0^h,	Chro. cor.	$+1^h\ 16^m\ 25^s.0$
	$-7^s.65 \times 11.25 =$	$-1\quad 26\ .1$
Jan. 26, 6^h	Chro. cor.	$+1\quad 14\quad 58\ .9$

This chronometer is *slow* and *gaining*.

2. To find the chro. cor. to reduce to local time, Jan. 26, 0^h, in long. 85° 16′ E.

Chro. cor. (Jan. 26 6^h G. t.)		$+1^h\ 14^m\ 58^s.9$
—Long.	$+6$	$+5\quad 41\quad 4$
Red. for	-12	$+3\ .8$
Chro. cor. (Jan. 26 0 L. t.)		$+6\quad 56\quad 6\ .7$ or $-5^h\ 3^m\ 53^s.3$

3. To form a table of chronometer correction for each day from Jan. 26, 6^h to Feb. 6, 6^h.

G. m. t.	*Chro. cor.*	*G. m. t.*	*Chro. cor.*
Jan. 26 6^h	$+1^h\ 14^m\ 58^s.9$	Feb. 1 6^h	$+1^h\ 14^m\ 13^s.0$
27 6	14 51 .3	2 6	14 5 .4
28 6	14 43 .6	3 6	13 57 .7
29 6	14 36 .0	4 6	13 50 .1
30 6	14 28 .3	5 6	13 42 .4
31 6	+1 14 20 .7	6 6	+1 13 34 .8

169. To find the *rate*, or *daily change*, of a chronometer, it is necessary to find the correction of the chronometer on two different days, either from observations, or by compari-

son with a chronometer, whose correction is known. Let c_1 and c_2 be the two corrections, t the interval expressed in days; then we have for the daily change,

$$\varDelta c = \frac{c_2 - c_1}{t}; \tag{134}$$

that is, the daily change is equal to the difference of the two chronometer corrections divided by the number of days and parts in the interval. If attention is paid to the signs, + will indicate that the chronometer is *losing*, — that it is *gaining*.

EXAMPLES.

	Chro. 1615	*Chro.* 4872	*Chro.* 796
	h m s	h m s	h m s
Chro. cor. April 15 0^h	+0 18 16.2	—1 15 27.5	+0 0 16.6
" " " 27 8	+0 18 29.6	—1 14 58.6	—0 0 5.3
Change in 12.3 days,	+13.4	+28.9	—21.9
Daily change of cor.	+1.09	+2.35	—2.71

At fixed observatories an interval of one day may suffice. For rating sea-chronometers by observations made with a sextant and artificial horizon, an interval of from 5 to 15 days is desirable.

The sea-rate of a chronometer is sometimes different from its rate on shore, or even from its rate while on board ship in port. Some chronometers are affected by magnetic influences, so that their rates are varied by changing the direction of the XII. hour mark to different points of the horizon. All are slightly affected by changes of temperature, as perfect compensation is rarely attainable. The excellence of a chronometer depends upon the permanence of its rate. The rate may be large, but if its variations are small, the chronometer is good.

170. A watch is often used for noting the time of an observation. It is compared with the chronometer by noting

the time of each at the same instant. The most favorable instant is when the watch shows no 0^s.

Let C and W be these noted times; then $\Delta W = (C - W)$ is the reduction of the watch time to the chronometer time: for $C = W + (C - W)$.

Comparisons should be made before and after the observation, and the results interpolated to the time of observation.

A practised observer may, by looking at the watch and counting the beats of the chronometer, make the comparison to the nearest $0^s.25$. It is better to take the mean of several comparisons than to trust to a single one.

A mean time and a sidereal chronometer may be compared within $0^s.03$ by watching for the coïncidence of beats, which occurs at intervals of 3^m, for chronometers, which beat half-seconds.

EXAMPLES.

	Chro. 476	*Chro.* 4072	*Chro.* 1976	*Chro.* 1976
	h m s	h m s	h m s	h m s
Chro.	4 16 56.2	3 15 17.5	11 48 18.2	1 0 28.5
Watch	1 5 0	7 35 30	3 16 0	4 28 0
$C - W$.	+3 11 56.2	—4 20 12.5	—3 27 41.8	—3 27 31.5

The last two are comparisons of the watch with the same chronometer. Suppose the time of an observation as noted by the watch to be $3^h\ 37^m\ 17^s$; for finding the corresponding time by the chronometer we have,

The change of $C - W$ in $1^h.2$, $+10^s.3$;
whence the change in 1^h is $+\ 8\ .6$,
and the change in $21^m.3 = 0^h.35$, the interval between the 1st comparison and the observation, $+3^s.0$;
or, by proportion, we have

$$72^m : 21^m.3 = +10^s.3 : +3^s.0$$

Then,	Time by Watch $=$	$3^h\ 37^m\ 17^s$
	$C - W =$	$-\ 3\ \ 27\ .\ 38.8$
	Time by chro. $=$	$0\ \ 9\ \ 38.2$

171. Problem 49. *To find the correction of a chronometer at a place whose latitude and longitude are given.*

1*st Method.* (By *single* altitudes.)

Observe an altitude, or set of altitudes, of the sun or a star, noting the time by the chronometer, or a watch compared with it.

Find from the altitude (Prob. 43) the local mean, or sidereal, time, as may be required.

The "local time" — the "chronometer time," or

$$c = T - C$$

(art. 135), is the correction of the chronometer on local time. Applying to this the known longitude of the place of observation, gives the correction on Greenwich time.

172. If an artificial horizon is used, as it should be when practicable, it is best to make two sets of observations with the roof in reversed positions. In A. M. observations of the sun with a sextant and artificial horizon, the lower limb of the sun and the upper limb of its image in the horizon are made to lap, and the instant of separation is watched for; while in P. M. observations the limbs are separated and approaching, and the instant of contact is noted. In observations of the upper limb this is reversed. Even a good observer may estimate the contact of two disks differently when they are separating and when they are approaching. Both limbs, then, should be observed.

In observing altitudes which change rapidly it is better, when circumstances permit, to set the instrument so as to read exact divisions at regular intervals, and watch the instant of contact. A good observer, with a sextant and artificial horizon, can observe the double altitudes at regular intervals of 10′.

173. On a subsequent day repeat this observation, and find again the correction of the chronometer. The differ-

ence between these two corrections divided by the number of days and parts in the interval is the *daily change*, as in Art. 169.

It is important that both the observations thus compared should be at nearly the same altitude and on the same side of the meridian (when the sun is observed, both in the forenoon, or both in the afternoon), and in general, that they should be made with the same instruments, and as nearly as practicable under the same circumstances. Thus, an error in the assumed latitude and *constant* errors of the instruments or the observer will affect the two chronometer corrections nearly alike, but will very slightly affect their difference, and, consequently, the rate determined from it will be nearly exact. The chronometer correction, derived from single altitudes, may be erroneous a few seconds. But for sea chronometers this is of less importance than an erroneous determination of the rate. For instance, suppose the determined chronometer correction in error 4^s, and the daily change in error 1^s; in 20 days (Art. 168) the computed change of the correction will be in error 20^s, and in 30 days will be in error 30^s.

174. 2*d Method.* (By *double* altitudes.)

It is better to observe altitudes of the body on both sides of the meridian, and as nearly at the same altitude as practicable, either on the same day or on two consecutive days.

Altitudes of two stars also may be used, one east, the other west of the meridian.

The mean of the two results is better than a determination from either alone; for constant errors of the latitude, the instrument, or the observer, affect the two results in opposite directions; that is, if one result is too large, the other is too small, and by nearly the same amount.

EXAMPLES. (Prob. 49.)

1. *Chronometer Correction.*

Pensacola Navy-Yard, 30° 20′ 30″ N., 87° 15′ 21″ W. 1865, May 30 21^h; Chro. 1876.

T. by Chro.		*Sextant No. 2.*	*Art. Hor. No. 1.*	
m s	s	° ′		m s
31 41		2 $\overline{\odot}$ 99 50 *A. end.*	Chro. cor. (G. m. t.)	—42 26
	22.7		Daily change	— 3.8
32 3.7		100 0		
	23.3		⊙'s diam. off arc	+32′ 8.3″
32 27		100 10	on arc	—30 59.2
	24			
32 51		100 20	In. cor.	+ 34.5
	23			
33 14		100 30		
	23.7			
33 37.7		100 40		
34 7.5		2 $\underline{\odot}$ 99 50 *B. end.*	Bar. 30.14	
	23			
34 30.5		100 0	Ther. 76°	
	23			
34 53.5		100 10		
	23.3			
35 16.8		100 20		
	23			
35 39.8		100 30		
	23.2			
36 3		100 40		
3 32 39.07		2 $\overline{\odot}$ 100 15		
3 35 5.18		2 $\underline{\odot}$ 100 15		

Computation.

	h m s	*⊙'s. dec.*		*Eq'n of t.*	
T. by Chro.	3 32 39.07	° ′ ″	″	m s	s
Chro. cor.	— 42 26	+21 57 40.3	+21.04	+2 36.35	—0.360
			42.08		.720
G. m. t. May 31	2 50 11	+ 59.7	16.83	— 1.02	.288
			.62		11
	2.837	+21 58 40.0	.15	+2 35.33	2

			$\overline{\odot}$ 50° 7′ 30″	½ In. cor.	
	h m s		— 16 17.3	+17.3″	ref. —51.6″
				S. diam.	
L. ap. t. May 30	21 3 51.35	$h =$	49 51 12.7	—15′ 48.4	par. +5.4
—Eq. of t.	— 2 35.33	$L =$	30 20 30	l. sec	0.0639749
L. m. t. May 30	21 1 16.02	$p =$	68 1 20.0	l. cosec	0.0327661
T. by Chro.	3 32 39.07	$2s =$	148 13 2.7		
$\overline{\odot}$, Chro. cor. (L.m.t)	—6 31 23.05	$s =$	74 6 31.4	l. cos	9.4374537
		$s—h =$	24 15 18.7	l. sin	9.6136320
					19.1478267
		$\frac{1}{2}t =$	157° 58′ 55.1″*	l. sin	9.5739134
		$t =$	315 57 50.2		

T. by Chro.	3h 35m 5.18s	⊙'s *dec.*		*Eq'n of t.*	
Chro. cor.	— 42 26	+21° 58′ 40.0″	+21.04″	+2m 35.33s	—0.360s
G. m. t. May 31	2 52 41	in 0h.041 + .8		— .02	
	2.878	+21 58 40.8		+2 35.31	

			$\underline{\odot}$ 50° 7′ 30″	½ In. cor.	
	h m s		+ 15 19.5	+17.3″	ref. —51.6″
				S. diam.	
L. ap. t. May 30	21 6 17.89	$h =$	50 22 49.5	+15′ 48.4	par. +5.4
—Eq. of t.	— 2 35.31	$L =$	30 20 30	l. sec	0.0639749
L. m. t. May 30	21 3 42.58	$p =$	68 1 19.2	l. cosec	0.0327668
T. by Chro.	3 35 5.18	$2s =$	148 44 38.7		
$\underline{\odot}$, Chro. cor. (L.m.t.)	—6 31 32.60	$s =$	74 22 19.4	l. cos	9.4303807
		$s—h =$	23 59 29.9	l. sin	9.6091709
					19.1362933
Mean	—6 31 22.82	$\frac{1}{2}t =$	158° 17′ 14.2″	l. sin	9.5681467
Red for 3h.0	— .48	$t =$	316 34 28.4		
Chro. cor. (L.m.t.)	—6 31 23.30	May 31 0h			

2. *Chronometer Correction.*

Pensacola Navy-Yard, 30° 20′ 30″ N., 87° 15′ 21″ W. 1865, May 31 3h.

* In A. M. observations, ½ t may be taken in the 2d quadrant; or it may be taken in the 1st quadrant and marked —.

T. by Chro.		*Sextant No. 2.*	*Art. Hor. No. 1.*
9h 24m 2.7s		2 $\overline{\odot}$ 100° 40′ *A. end.*	Chro. cor. (G. m. t.) —42m 27s
	22.8s		Daily change — 3.8
24 25.5		100 30	
	23.0		$\odot$'s diam. off arc +32′ 12.5″
24 48.5		100 20	on arc —30 59.2
	24.0		
25 12.5		100 10	In. cor. + 36.6
	22.3		
25 34.8		100 0	
	23.4		
25 58.2		99 50	
28 33.5		2 $\underline{\odot}$ 97 40 *B. end.*	Bar. 30.14
	23.5		
28 57		97 30	Ther. 76°
	23.5		
29 20.5		97 20	
	22.5		
29 43		97 10	
	23.0		
30 6		97 0	
	23.5		
30 29.5		96 50	
9 25 0.37		2 $\overline{\odot}$ 100 15	
9 29 31.58		2 $\underline{\odot}$ 97 15	

Computation.

	h m s	*$\odot$'s dec.*		*Eq'n of t.*	
T. by Chro.	9 25 0.37				
Chro. cor.	— 42 27	+21° 57′ 40.3″	+20.92″	+2m 36.35s	—0.362s
			167.36		2.896
G. m. t. May 31	8 42 33	+3 2.2	14.64	—3.15	.253
	8.709	+22 0 42.5	19	+2 33.20	·3

		$\overline{\odot}$ 50° 7′ 30″	½ In. cor. +18.3″ ref. —51.6″
		—16 16.3	
		h= 49 51 13.7	S. diam. +15′ 48.4 par. +5.4
		L= 30 20 30	l. sec 0.0639749
	h m s	p= 67 59 17.5	l. cosec 0.0328703
L. ap. t. May 31	2 56 12.04	2s=148 11 1.2	
—Eq. of t.	—2 33.20	s= 74 5 30.6	l. cos 9.4379032
L. m. t. May 31	2 53 38.84	$s-h$= 24 14 16.9	l. sin 9.6133431
T. by Chro.	9 25 0.37		19.1480915
$\overline{\odot}$, Chro. cor.(L.m.t.)	—6 31 21.53	½t= 22° 1′ 30.3″	l. sin 9.5740458
		t= 44 3 0.6	

	h m s	⊙'s *dec.*		*Eq'n of t.*	
T. by Chro.	9 29 31.58	° ′ ″	″	m s	s
Chro. cor.	−42 27	+22 0 42.5	+20.92	+2 33.20	−0.362
G. m. t. May 31	8 47 5	in $0^h.076$ +1.6		−.03	
	8.785	+22 0 44.1		+2 33.17	

		⊙	48° 37′ 30″	½ In. cor. +18″.3 ref. −54″.5
			+15 17.8	S. diam.
	h m s	h =	48 52 47.8	+15′ 48.4 par. +5.6
L. ap. t. May 31	3 0 42.95	L =	30 20 30	l. sec 0.0639749
−Eq. of t.	−2 33.17	p =	67 59 15.9	l. cosec 0.0328717
L. m. t. May 31	2 58 9.78	$2s$ =	147 12 33.7	
T. by Chro.	9 29 31.88	s =	73 36 16.8	l. cos 9.4506544
⊙, Chro. cor.(L.m.t.)	−6 31 21.80	s–h =	24 43 29.0	l. sin 9.6214453
			° ′ ″	19.1689463
(mean)	−6 31 21.66	$\frac{1}{2}t$ =	22 35 22.1	l. sin 9.5844732
Red for -3^h.	+.48	t =	45 10 44.2	
Chro.cor.(L.m.t.)	−6 31 21.18, May 31 0^h.			

May 31 0^h Chro. cor. (L. m. t.) $-6^h\ 31^m\ 22^s.24$ by A.M. and P.M. obs.
Long. +5 49 1 .4
May 31 6^h Chro. cor. (G. m. t.) − 42 20 .84

3. *Table of Chro. Corrections.*

Chro. 1876 ; *fast* of Greenwich mean time and *gaining.*

G. m. t.	*Chro. cor.*	*Daily Ch.*	*Remarks.*
h	h m s		
1865, May 1 3	−0 40 20.5		⊙, A.M. Key West Light-House.
		s −4.14	
17 3	41 26.8		⊙, A.M. " " " "
		3.88	
25 6	41 58.3		⊙, A.M. & P.M. Pensacola Navy-Yard.
		3.75	
31 6	42 20.8		⊙, A.M. & P.M. " " "

Long.* of Key West Light-House, 81° 48′ 40″ W.
Long. of Pensacola Navy-Yard, 87 15 21 W.

* The assumed longitudes of places, where the chronometer is rated, should be stated.

4. *Comparisons and Corrections of Chronometers.*

1865, May 31, 6^h, G. mean time.

	Chro. 4375	*Chro.* 9163	*Chro.* 789	*Chro.* 5165
	h m s	h m s	h m s	h m s
Chro.	6 50 16.3	5 3 29.7	2 15 27.5	11 59 16.8
(1876)	6 30 0	6 31 0	6 32 10	6 33 30
(1876)—Chro.	−0 20 16.3	+1 27 30.3	+4 16 42.5	−5 25 46.8
Cor. of (1876)	−42 20.8	−42 20.8	−42 20.8	−42 20.8
Chro. cor.	−1 2 37.1	−0 45 9.5	+3 34 21.7	−6 18 7.6
				or +5 41 52.4

175. 3*d Method.* (By *equal* altitudes.)

A heavenly body, which does not change its declination, is at the same altitude east and west of the meridian at the same interval of time from its meridian passage.

If, then, such equal altitudes are observed and the times noted by the chronometer, or by a watch and reduced to the chronometer (Art. 170), the mean of these times, or the *middle time*, is the chronometer time of the star's meridian transit.

The corresponding sidereal time is the star's right ascension, when the first observation is east of the meridian ; $12^h +$ the right ascension when the first observation is west of the meridian.

This, for a mean time chronometer, may be converted into local mean time (Prob. 32) ; and for a Greenwich chronometer into the corresponding Greenwich time.

Subtracting the chronometer time, we have the correction of the chronometer.

EXAMPLE.

1865, Jan. 14, at Washington, in longitude 77° 2′ 48″ W., equal altitudes of α Canis Minoris were observed, and the times noted by a chronometer regulated to Greenwich mean time ; from which were obtained :

Mean of chro. times (⁎ east)		2^h	16^m	$35^s.65$
" " " " (⁎ west)		7	59	16 .38
Chro. time of ⁎'s transit		5	7	56 .01
L. sid. t. = ⁎'s R. A.		7	32	16 .26
Long.		+5	8	11 .2
G. sid. t.		12	40	27 .46
$-S_0$	(Jan. 14)	−19	35	51 11
Sid. int. from Jan. 14 0^h		17	4	36 .35
Red. to m. t. int.			−2	47 .86
G. mean time	Jan. 14	17	1	48 .49
Chro. time		17	7	56 .01
Chro. cor.			−6	7 .52

176. If equal altitudes of the sun are observed in the forenoon and afternoon of the same day, the mean of the noted times would be the chronometer time of *apparent noon*, were it not for the change of the sun's declination between the observations.

Problem 50. *In equal altitudes of the sun, to find the correction of the middle time for the change of the sun's declination in the interval between the observations.*

Solution. Let

h = the sun's true altitude at each observation,

t = half the elapsed *apparent* time between the observations,

T_0 = the mean of the chronometer times of the two observations, or the *middle* chronometer time,

ΔT_0 = the correction of this mean to reduce to the chronometer time of apparent noon;

L = the latitude of the place,

d = the sun's declination at local apparent noon,

Δd = the change of this declination in the time t;

then, when both observations are on the same day,

$t + \Delta T_0$ will be numerically the hour-angle at the A. M. observation,

$t - \Delta T_0$, the hour-angle at the P. M. observation,

$d - \Delta d$, the declination* at the A. M. observation,
$d + \Delta d$, the declination* at the P. M. observation.

By (116), we have for the two observations,

$$\left.\begin{array}{l}\sin h = \sin L \sin (d - \Delta d) + \cos L \cos (d - \Delta d) \cos (t + \Delta T_0) \\ \sin h = \sin L \sin (d + \Delta d) + \cos L \cos (d + \Delta d) \cos (t - \Delta T_0)\end{array}\right\} \quad (135)$$

But

$$\sin (d \pm \Delta d) = \sin d \cos \Delta d \pm \cos d \sin \Delta d,$$
$$\cos (d \pm \Delta d) = \cos d \cos \Delta d \mp \sin d \sin \Delta d,$$
$$\cos (d \pm \Delta T_0) = \cos t \cos \Delta T_0 \mp \sin t \sin \Delta T_0.$$

Since Δd, and therefore ΔT_0, are very small, we may put

$$\cos \Delta d = 1, \qquad \sin \Delta d = \Delta d. \sin 1'',$$
$$\cos \Delta T_0 = 1, \qquad \sin \Delta T_0 = 15\, \Delta T_0. \sin 1'';$$

Δd being expressed in seconds of arc, and
ΔT_0 in seconds of time; we shall then have

$$\sin (d \pm \Delta d) = \sin d \pm \Delta d. \sin 1'' \cos d,$$
$$\cos (d \pm \Delta d) = \cos d \mp \Delta d. \sin 1'' \sin d,$$
$$\cos (t \pm \Delta T_0) = \cos t \mp 15\, \Delta T_0. \sin 1'' \sin t.$$

Substituting these in the two equations (135), subtracting the first from the second, and dividing by 2 sin 1″, we shall have

$$0 = \Delta d. \sin L \cos d - \Delta d. \cos L \sin d \cos t$$
$$+ 15\, \Delta T_0. \cos L \cos d \sin t.$$

Transposing and dividing by the coëfficient of ΔT_0, we find the formula

$$\Delta T_0 = -\frac{\Delta d. \tan L}{15 \sin t} + \frac{\Delta d. \tan d}{15 \tan t}, \qquad (136)$$

which is called the equation of equal altitudes.

Let

$\Delta_h d =$ the hourly change of declination at the instant of apparent noon, and express
t, which is half the elapsed apparent time, in hours,

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* Strictly, in the one case, Δd should be the change of declination in the time $t + \Delta T_0$; in the other, the change in the time $t - \Delta T_0$.

then $$\Delta d = \Delta_h d.\ t,$$

and (136) becomes

$$\Delta T_0 = -\frac{\Delta_h d.\ t \tan L}{15 \sin t} + \frac{\Delta_h d.\ t \tan d}{15 \tan t}. \tag{137}$$

If we put

$$A = -\frac{t}{15 \sin t}, \quad B = \frac{t}{15 \tan t} \tag{138}$$

and

C_0 = the chronometer time of apparent noon, we have

$$\left.\begin{aligned} \Delta T_0 &= A.\ \Delta_h d. \tan L + B.\ \Delta_h d. \tan d \\ C_0 &= T_0 + \Delta T_0 \end{aligned}\right\} \tag{139}$$

In these formulas, L and d are + when *north*, Δd and $\Delta_h d$ are + when the sun is moving toward the *north*.

The coëfficient A is —, since $t < 12^h$,

" " B is + when $t < 6^h$, — when $t > 6^h$.

The computation of the two parts of ΔT_0 is facilitated by tables of log A and log B. Such tables are given in Chauvenet's "Method of finding the error and rate of a chronometer," in the American Ephemeris and Nautical Almanac for 1856, and reprinted in a pamphlet with his "New method of correcting Lunar distances."

The argument of these tables is 2 t, or the elapsed time. The signs of A and B are given.

Apply the two parts of ΔT_0, according to their signs, to the *Middle Chronometer Time;* the result is the *Chronometer Time of Apparent Noon.*

Apply to this the equation of time (*adding*, when the equation of time is *additive*, to *mean* time; otherwise *subtracting*); the result is the *Chronometer Time of Mean Noon* at the place.

Applying to this the longitude (in time), *subtracting* if *west*, *adding* if *east*, gives the *Chronometer Time of Mean Noon at Greenwich.*

*12^h — *Chro. T. at local Mean Noon*, will be the *Chro. correction* if the chronometer is regulated to local time.

*12^h — *Chro. T. at Greenwich Mean Noon*, will be the *Chro. correction*, if the chronometer is regulated to Greenwich time.

177. If a set of altitudes is observed in the afternoon of one day, and a set of equal altitudes in the forenoon of the next day, the middle time would correspond nearly to the instant of *apparent midnight;* and half the elapsed time t, would be nearly the hour-angle from the lower branch of the meridian, or the supplement of the proper hour-angle.

In this case

$180° - (t + \varDelta T_0)$	will be the hour-angle at the	P.M.	observation.
$180° - (t - \varDelta T_0)$	" " " " " "	A.M.	"
$d - \varDelta d$,	the declination at the	P.M.	"
$d + \varDelta d$,	" " " "	A.M.	"

and we have for the two observations, as in (135)

$$\left.\begin{array}{l}\sin h = \sin L \sin (d - \varDelta d) - \cos L \cos (d - \varDelta d) \cos (t + \varDelta T_0) \\ \sin h = \sin L \sin (d + \varDelta d) - \cos L \cos (d + \varDelta d) \cos (t - \varDelta T_0)\end{array}\right\} \dagger(140)$$

Treating these in the same way as (135) we shall have

$$0 = \varDelta d . \sin L \cos d + \varDelta d . \cos L \sin d \cos t$$
$$- 15 \varDelta T_0 \cos L \cos d \sin t;$$

whence

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* This is better noted as 0^h.

† These may be written

$$-\sin h = - \sin L \sin (d - \varDelta d) + \cos L \cos (d - \varDelta d) \cos (t + \varDelta T_0)$$
$$-\sin h = - \sin L \sin (d + \varDelta d) + \cos L \cos (d + \varDelta d) \cos (t - \varDelta T_0).$$

They differ from (130) in the signs of h and L and in reckoning the hour-angles from the lower, instead of the upper, branch of the meridian. This would be the case, if we suppose the observations to be referred to the latitude and meridian of the antipode. The only effect in (136) is to change the sign of tan L, or of the first term in the equation of equal altitudes.

$$\Delta\, T_0 = \frac{\Delta\, d.\ \tan L}{15 \sin t} + \frac{\Delta\, d.\ \tan d}{15 \tan t}$$

or, putting as before $\Delta\, d = \Delta_h\, d.\ t$

$$A = -\frac{t}{15 \sin t}, \quad B = \frac{t}{15 \tan t},$$

$$\Delta\, T_0 = -A.\ \Delta_h\, d.\ \tan L + B.\ \Delta_h\, d.\ \tan d, \qquad (141)$$

which differs from (139) only in the sign of A. This is the reduction of the middle time to the *Chro. Time of apparent midnight:* applying the equation of time reduces it to the *Chro. Time of mean midnight.*

178. d, $\Delta'\, d$, and the equation of time are to be taken from the Almanac for the instant of apparent noon, or of apparent midnight, according as the observations are made on the same day, or on consecutive days.

$2\, t$ is properly the *elapsed apparent time.* The elapsed time by chronometer requires, then, not only a correction for the rate, which is

$$\frac{2\, t}{24^h} \Delta\, c, \text{ (+ when the chronometer loses)};$$

but also a reduction to an apparent time interval, which, for a mean time chronometer, is the change* of the equation of time in the time, $2\, t$, additive when the equation of time is additive to *mean* time and increasing, or subtractive from mean time and decreasing. For a sidereal chronometer, it is the change in the sun's right ascension in the time $2\, t$, and subtractive.

179. Equal altitudes of the moon or a planet may be observed; but in the case of the moon admit of less precision than of the sun, and moreover require correction for the inequality produced by change of parallax.

If $2\, \Delta\, a$ is the *increase* of right ascension in the interval,

* The maximum daily change is 30^s. The elapsed time by Chronometer is usually regarded as sufficiently accurate.

the body will arrive at its second position later than would a fixed star, supposed coïncident with it at the first position; and the elapsed *sidereal* time will be greater than the double hour-angle of the body by the quantity $2\ \varDelta\ a$. If $2\ s =$ the elapsed *sidereal* time, then in (137) we must take

$$2\ t = 2\ s - 2\ \varDelta\ a, \text{ or } t = s - \varDelta\ a. \tag{142}$$

If $t_m =$ half the elapsed *mean* time (expressed in hours when used as a coëfficient), and

$\varDelta_h\ a =$ the *increase* of right ascension in 1^h of *mean* time,

by (87) $$s = t_m + 9^s.8565\ t_m$$

and $$t = t_m + t_m\ (9^s.8565 - \varDelta_h\ a), \tag{143}$$

by which t and $2\ t$ may be found from $2\ t_m$ the elapsed mean time.

In this expression the last two terms are in seconds. Reducing to hours we have

$$t + t_m\left(1 = \frac{9^s.8565 - \varDelta_h\ a}{3600}\right) = t_m\left(1.002738 - \frac{\varDelta_h\ a}{3600}\right) \tag{144}$$

If $\varDelta_h\ d =$ the change of declination in 1_h of mean time, then in (136)

$$\varDelta\ d = t_m.\ \varDelta_h\ d$$

or, substituting for t_m its value from (144),

$$\varDelta\ d = t.\ \varDelta_h\ d \div \left(1.002738 - \frac{\varDelta_h\ a}{3600}\right)$$

Equations (138) and (139) may then be used for other bodies than the sun, provided we give t its proper value from (142) or (143), and for $\varDelta_h\ d$ substitute

$$\varDelta'_h\ d = \varDelta_h\ d \div \left(1.002738 - \frac{\varDelta_h\ a}{3600}\right),$$

or, which will be sufficiently exact,

$$\varDelta'_h\ d = \varDelta_h\ d + \frac{\varDelta_h\ a - 9^s.856}{3600}.\ \varDelta_h\ d \tag{145}$$

180. Observing the double altitudes at regular intervals of 10′, or 20′, especially facilitates the method of equal altitudes; for, if the first set is observed at equal intervals, in the second the observer, having set the instrument for the *last* reading of the first and observed the contact, for the subsequent observations, has only to move back successively the same intervals.

181. It is not requisite that the instrument should give the true altitude; it is sufficient if the altitude is the *same* at the two corresponding observations. Hence the two observations should be made with the same instruments, without change of adjustment, and in general as nearly as practicable under the same circumstances.

This purpose is promoted by making the final movement of the tangent screw in both sets always in the same direction. Thus, in reversing the movement, the screw may be turned a little too far, and then the final contact made by a motion in the same direction as before.

If the sun is used, both limbs should be observed.

The error arising from want of parallelism of the surfaces of the roof-glasses of the horizon is eliminated by having the *same* end of the roof toward the observer. The roof may be tested by observing sets of altitudes with it in reversed positions.

182. Although the readings of the instrument may be the same in the two sets of observations, the altitudes may be slightly different, 1st, from changes in the instrument in the interval; 2d, from difference of refraction at the two times.

A change in the index correction may be detected by observation; but there may be expansion or contraction of various parts of the instrument which may affect the readings of the altitudes without altering the index correction.

The change of refraction may be found by noting the ba-

rometer and thermometer at each set, and finding the refraction for both sets of altitudes.

183. To correct the middle time for any small difference of the altitudes, whether from refraction or actual change of readings, we may find, from the difference between two readings and the difference of the corresponding times, the change of time for a change of 1′, or 1″, of altitude. This multiplied by half the inequality of altitudes, expressed in minutes, or seconds, will give the correction of the middle time, to be added when the P. M. altitude is the greater; to be subtracted when the P. M. altitude is the less.

If twice the altitude is observed with an artificial horizon, we may find the change of time for a change of 1′, or 1″, of the double altitude, and multiply it by the whole inequality of the altitudes.

Examples. (Prob. 50.)

1. 1865, Jan. 10 $9\frac{1}{2}^{h}$ A. M. and $2\frac{1}{2}^{h}$ P. M. Equal altitudes of ⊙ at the Custom-House, Key West, Florida; 24° 33′ 20″ N., 81° 48′ 37″ W. chro. 1085; chro. cor. (G. m. t.) $-42^{m}\ 18^{s}.0$; daily change $+8^{s}.3$.

Sex. No. 1. *Art. Hor. No.* 2.	*T. by Chro.* *A.M.*	*P.M.*	*Mid. time.* $6^{h}\ 17^{m}$		*A.M.*	*P.M.*
° ′	h m s	h m s	s			
2⊙ 60 0 *A. end.*	3 39 26.7	8 55 59.7	43.2	⊙'s *diam.*	+32′ 25″.0	+32′ 26″.7
10	39 58.8	55 27.0	42.9		—32 41 .7	—32 43 .3
20	40 31.5	54 55.3	43.4	In. cor.	—8 .3	—8 .3
30	41 4.0	54 22.0	43.0			
40	41 35.7	53 50.3	43.0	Bar.	30.22	30.18
50	42 9.0	53 16.5	42.8	Ther.	77°	80°
2⊙ 60 0	42 58.5	52 26.7	42.6			
10	43 31.3	51 53.5	42.4	Ref.	—1′ 34″.9	—1′ 34″.4
20	44 3.8	51 21.0	42.4	Diff. of alt.		$\Delta h = +0.5$
30	44 37.5	50 48.0	42.7			
40	45 10.0	50 15.5	42.8	For $2\,\Delta h = 10'$,		$\Delta t = 32^{s}.8$
50	45 43.7	49 42.0	42.8	$2\,\Delta h = 1''$,		$\Delta t = 0.055$
60 25	3 42 34.21	8 52 51.46				

Elapsed Chro. t.	h m s 5 10 17.25	Long. +5h 27m 14s.5		Eq. of t +7m 56s.96	+0s.998
Mid. Chro. t.	6 17 42.83	5h.456		+5. 45	4.990
Red. for Δh, 0s.055 × 0.5 = +.03		0d.227		+8 2.41	456
1st part of Eq.	—2.89	☉'s dec. —21° 54′ 43″		$\Delta_h d$ +22″.78	ch. in 1d +1″.065
2d " " "	—1.93	+2 5		+.24	.213
Chro. t. of ap. noon	6 17 37.99	—21 52 38		+23 .02	21 7
—Eq. of time	— 8 2.41	L' = +24° 33′.3	t. tan 9.6598	d = —21° 52′.6	l. tan 9.6037 n
Chro. t. of m. noon	6 9 35.58	$\Delta_h d$ = +23″.02	log 1.3621		log 1.3621
—Long.	—5 27 14.5	A	log 9.4396 n	B	log 9.3315
Chro.t.of G.m. noon	0 42 21.08	—2s.89	log 0.4615 n	—1s.98	log 0.2973 n
Chro. cor. (G. m.t)	—42 21.08	Jan 10 5h.5			

(The elapsed apparent time is $5^h\ 10^m\ 12^s$.)

2. 1865, June 19, $4\frac{1}{2}^h$ P.M., and 20, $7\frac{1}{2}$ A.M.; nearly equal alt. of ☉; at Belize, S. E. pass of Mississippi River, 29° 7′ 8″ N., 89° 5′ 18″ W., chro. 1085; chro. cor. (G. m. t.) $-41^m\ 28^s$; daily change $+1^s.0$; sextant No. 2; art. hor. No. 1; (A. end toward observer).

P.M.		*A.M.*			*P.M.*	*A.M.*
Sex. Read.	*T. by Chro.*	*T. by Chro.*	*Sex. Read.*	In. cor.	+41″.8	+40″.4
° ′	h m s	h m s	° ′			
$2\underline{\odot}$ 65 10	10 55 48	2 22 44.5	$2\underline{\odot}$ 65 30	Bar.	30 .09	30 .12
65 0	56 10.8	22 21	65 20	Ther.	81°	80°
64 50	56 34.3	21 34	65 0	½ In. cor.	+ ′ 20″.9	+ ′ 20″.2
64 40	56 57.5	21 11.2	64 50	Ref.	—1 28 .7	—1 28 .5
					—1 7 .8	—1 8 .3
$2\overline{\odot}$ 65 30	57 27.5	20 41.5	$2\overline{\odot}$ 65 40	(P.M.)—(A.M.)		
65 20	57 50.5	20 18	65 30	Diff. ref., &c.		+ ′ 0″.5
65 10	58 14.3	19 54	65 20	Diff. obs. alts.		—5 37 .5
65 0	58 37.3	19 7.5	65 0		Δh =	—5 37 .0
2☉ 65 5 0	10 57 12.53	2 20 58.96	2☉ 65 16 15	For $2\Delta h$=10′,		Δt=23s.3
h'=32 32 30			h'=32 38 7.5	$2\Delta h$= 1′,		Δt= 2 .33

G. ap. t., June 19, $17^h\ 56^m\ 21^s.2 = 19,\ 17^h.939 = 19^d.747$

☉'s *dec.*	Δ_h *d.*	*Ch. in* 1d.	*Eq. of t.*	
° ′ ″	″	″	m s	s
+23 26 28	+2.01	—1.03	+0 59.99	+0.543
+29	—.77	.72 5		5.43
			+9.74	3.801
+23 26 57	+1.24			.489
			+1 9.73	21

	h	m	s				
Middle Chro. t.*	18	39	5.75	Elapsed T. by Chro.*		$15^h\ 23^m\ 46^s$	
Red. for Δh, $-5'.62 \times 2^s.33 =$			-13.09	. . . .		+ 26	
1st part of Eq.			+.39	Ch. of Eq. of t.		— 8	
2d " " "			—.13	Elapsed ap. t.		15 24 4	
Chro. t. of ap. 12^h	18	38	52.92	$L = +29°\ 7'.1$	l. tan 9.7459	$d = +23°\ 27'$	l. tan 9.6373
—Eq'n of t.		—1	9.73	$\Delta_h d = +1''.24$	log 0.0934	. . . .	log 0.0934
Chro. t. of mean 12^h	18	37	43.19	A	log 9.7550	B	log 9.3891 n
—Long.	—5	56	21.2	$+0^s.39$	log 9.5943	$-0^s.13$	log 9.1198 n
Chro. t. of G. mean 12^h	12	41	21.99				
Chro. cor. (G. m. t.)		—41	21.99	June 19 18^h			

184. *4th method* of finding the correction of a chronometer. (By *transits.*)

On shore, the most accurate method of finding the correction of a chronometer is by noting the times of transit of the sun or a star across the threads of a well-adjusted transit instrument. The mean of these times is taken and corrected for the errors of the instrument, or reduced to the meridian. In the case of the sun, the transits of both limbs may be observed; or only one, and the "sidereal time of the semidiameter passing the meridian," found on page I. of each month in the almanac, added for the limb, which transits first; subtracted for the second limb.

At the instant of a star's transit of the meridian, the right ascension of the star is the *sidereal* time. The instant of transit of the sun's centre is *apparent* noon.

From either of these, the local *sidereal* or *mean* time, as may be required, can be found; and thence the chronometer correction by subtracting the chronometer time of transit.

The moon should not be used for finding the time, when precision is required. Stars are preferred to the sun, either when transits are observed, or equal altitudes with the artificial horizon; chiefly because many stars may be observed

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* To obtain these, 24^h was added to the P. M. chro. time. Twice the reduction of the middle time for the diff. of alts. is to be added to the elapsed time when the P. M. observation is last; subtracted when the P. M. observation is first. This may be neglected unless the diff. of altitudes is quite large.

during the same night, and the instrument is not exposed to the rays of the sun.

185. By repeating the transits on a subsequent day, the chronometer correction can be again found, and from the two corrections, the rate as in Art. 169. If the transit instrument is not well adjusted, or the instrumental corrections are imperfectly known, the *rate* of the chronometer can still be quite well determined from transits of the same star, or the same set of stars, on different days, provided the position of the instrument, or its adjustments, have not been disturbed in the interval.

186. A rough substitute for a transit instrument is a vertical corner of a building, and a position for the eye in its meridian. The instant of the appearance or disappearance of a star, or a limb of the sun, may be noted by a chronometer, and the chronometer correction obtained as with a transit instrument; but with much less accuracy, since the mode of observing is rough, and the position for the eye cannot be adjusted to the meridian with much precision: still the rate may be found with tolerable accuracy from the transits of the same body on different days.

LONGITUDE.

187. To find the longitude of a place by astronomical observations, it is generally necessary to determine independently the local and Greenwich times of the same instant. The difference of these times is the longitude, which is *west* when the Greenwich time is the greater, and *east* when the Greenwich is the less (Art. 165). This is expressed by (72)

$$\lambda = T_0 - T,$$

in which T_0 is the Greenwich time, and

T, the corresponding local time of the same kind.

These times may be *apparent*, *mean*, or *sidereal*.

The apparent time is the hour-angle of the true sun; the mean time, that of the mean sun; the sidereal time, that of the vernal equinox. In the same way we may use the local and Greenwich hour-angles of any other body or point of the heavens, regarded as + toward the *west.*

This is evident from Fig. 35; for if

P M is the meridian of Greenwich,
P M′, the local meridian,
P S, the declination circle of a heavenly body;
M P M′ will be the longitude of the place,
M P S, the hour-angle of the body at Greenwich,
M′ P S, the local hour-angle;

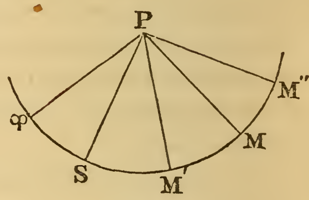

Fig. 35.

and we shall have, as in Art. 74,

$$M\,P\,M' = M\,P\,S - M'\,P\,S.$$

The several methods of finding the longitude differ in the modes of finding and comparing the two times, or the two hour-angles.

188. Problem 51. *To find the longitude of a place by a portable chronometer regulated to Greenwich time.*

Solution. The correction and rate of the chronometer are supposed to have been found by suitable observations at a place whose longitude is known. Let the chronometer be transported to the place whose longitude is required; and let an observation suitable for finding the hour-angle of a heavenly body, or the local time, be made and the time noted by the chronometer, or by a watch compared with it.

There are then two parts of the process to be pursued: 1st, from the noted time to find the Greenwich time (mean, apparent, or sidereal), or the hour-angle of the body, as may be deemed most convenient. 2d, from the observations, to find the corresponding local time, or hour-angle. Subtract-

ing the local time, or hour-angle, from the Greenwich time, or hour-angle, will give the longitude.

189. 1st. *To find the Greenwich time, or hour-angle, of the body observed*, apply to the noted time the reduction of the watch time to chronometer time, $C - W$ (if a watch has been used), and the chronometer correction, c', reduced to the date of observation (Art. 168).

The result is, the Greenwich time; and will be *mean* or *sidereal*, according as the chronometer is regulated to mean or sidereal time.*

If it is sidereal time, it will be necessary to reduce it to mean time (Prob. 32), except when a fixed star has been observed, so as to take from the almanac the quantities which will be required.

If, now, the Greenwich *hour-angle* of the body observed is desired:

In the case of the sun, reduce the Greenwich mean time to apparent time, by applying the equation of time.

If some other body has been observed, reduce the Greenwich mean time to sidereal time by adding the right ascension of the mean sun; and thence find the hour-angle of the body, by subtracting its right ascension. Or, if a sidereal chronometer has been used, from the Greenwich sidereal time subtract the right ascension of the body.

Attention to the signs will give the hour-angle thus obtained, + if toward the *west*, − if toward the *east*.

190. The Greenwich time or hour-angle is affected by the error of the chronometer correction, which consists, 1st, of the error in its original determination, which includes any error of the assumed longitude of the place of rating; 2d, of the error arising from an erroneous rate. This last error is cumulative, increasing with the number of days from the

* For observations of stars, a sidereal chronometer is most convenient.

date, when the correction of the chronometer was found from observations.

191. The chronometer correction for the date of observation can be derived from subsequent as well as from prior determinations of it and its daily change. In finding the longitude of a place on shore, or of a shoal, both values should be obtained, when practicable, and combined by giving weights to each inversely proportional to its interval of time from the original determination. Thus, if c' and c'' are two such chronometer corrections, the first brought forward t' days, the second carried back t'' days, we may take as the mean value*

$$\frac{t'' c' + t' c''}{t' + t''},$$

or, in a form more convenient for computation,

$$c' + \frac{t' (c'' - c')}{t' + t''}.$$

For example, suppose that on Jan. 17, the chronometer correction brought forward from Jan. 1, is $-18^m\ 56^s.5$, and reduced back from Jan. 25, is $-19^m\ 3^s.4$; the value by the above formula will be

$$-18^m\ 56^s.5 + \frac{16 \times -6^s.9}{24} = -19^m\ 1^s.1.$$

Two longitudes may be combined in a similar way.

192. Reports of longitudes by chronometer are regarded as of but little value, unless the number of chronometers, the assumed longitude of the place, where the chronometer

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* This assumes that c' and c'' are derived from two chronometer corrections of equal weight, and consequently that the longitudes used in finding them are equally reliable. This may not be the case if the chronometer corrections were found from observations at two different places.

The student is referred to Chauvenet's Astronomy, Vol. I., pp. 317, &c., for the methods of allowing for changes in the rates and combining the results of several chronometers.

is rated, and the age of the rates are stated.* Strictly, the chronometer merely determines the difference of longitude between the two places, where the observations are made. This may be obtained by using the chronometer correction on the time of the place of rating, instead of the Greenwich time. It is preferable to report such differences rather than absolute longitudes.

193. 2*d. To find the hour-angle of the body, and thence the local time.*

1*st Method.* (Problem 43. By *single* altitudes.) Observe in quick succession several altitudes of the heavenly body, noting the time of each by the chronometer, or by a watch compared with it.

Take the mean of the noted times, and from it find the Greenwich mean time; for which take from the Almanac the declination of the body, its semidiameter and horizontal parallax when sensible, as well as the quantities required for finding the Greenwich hour-angle. (Art. 189.)

Take the mean of the readings of the instrument, with which the altitudes were measured, and from it find the *true* altitude of the centre of the body (Art. 118). With this and the known, or assumed, latitude of the place find the *local hour-angle* of the body by Problem 43.

This hour-angle, which for the sun is the *local apparent time*, subtracted from the corresponding Greenwich hour-angle already found, will give the longitude.

Or, the *local mean time* may be found from it, for the sun, by applying the equation of time; for other bodies, by adding the right ascension of the body, which will give the *local sidereal time*, and subtracting the right ascension of the mean sun (Prob. 37): and the local time subtracted from the corresponding Greenwich time will give the longitude.

* See U. S. Navy Regulations, 459.

194. On shore, it is best to use an artificial horizon, even when a sea-horizon can be had, and for precise observations, stars in preference to the sun.

At sea, the sun is most conveniently used; but altitudes of the moon and bright stars can be employed, when the sun is not available. The chief difficulty is the obscurity of the sea-horizon at night. During twilight, however, or in a bright moonlight, it is often distinct and well defined.

195. The most favorable position of the body for finding its hour-angle from its altitude is, as previously stated, when it is nearest the prime vertical; provided its altitude is not so small, as to involve to too great an extent the uncertainty of refraction; and, observed on shore, is within the limits* of the instruments employed.

On shore, the time and circumstances most favorable for observations can generally be selected. At sea, long continuance of bad weather may render poor observations, made under unfavorable circumstances, the only ones available.

While, then, it is not well to use for finding the time an altitude less than 10°, or of an object, whose azimuth is less than 45° or more than 135°, it may sometimes be necessary to exceed these limits.

196. When the declination and latitude are nearly the same, the body is nearest the prime vertical but a short time before and after its meridian passage, so that a very great altitude may be used. Thus in lat. 20° N., the sun, when its declination is 19° 55′ N. or 20° 5′ N., is nearest the prime vertical within 22^m of noon at an altitude of nearly 85°; and the local time can be as accurately obtained from an altitude of 89°, 4^m from noon, and about 5° in azimuth from the prime vertical, as from an altitude of 30°, provided the assumed latitude can be depended on within 2′. Nearer noon, the rapid change of the sun's azimuth, averaging 10°

* For a sextant and artificial horizon, between 20° and 60°.

in 1^m, would make it difficult to observe the altitude with sufficient precision.

197. The local time or hour-angle is affected by errors in the altitude and in the assumed latitude. (Arts. 136, 138.) When several observations have been made in rapid succession, the effect of a supposed error of $1'$ in the altitude* may be found by dividing the difference of two of the noted times by the difference, in *minutes*, of the corresponding altitudes.

In a similar way we may find the change of altitude in 1^m of time by dividing the difference of two altitudes by the difference in *minutes*, of the corresponding times. The maximum change of altitude in 1^m is $15'$; when $L = 0$ and $d = 0$. The more rapid the change of altitude, the less will errors of altitude affect the result.

To ascertain the effect of an error of $1'$ in the assumed latitude,† the local times or hour-angles may be computed separately for two latitudes differing $10'$, or $20'$, from each other, and the difference of these times divided by $10'$, or $20'$. At sea, the latitude by account is used, either brought

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* From the equations,

$$\sin h = \sin L \sin d + \cos L \cos d \cos t,$$
$$\sin (h + \varDelta h) = \sin L \sin d + \cos L \cos d \cos (t + \varDelta t),$$
$$\sin Z = \frac{\cos d \sin t}{\cos h},$$

we shall find by processes similar to those pursued in Art. 176, on the supposition that $\varDelta h$ and $\varDelta t$ are very small,

$$\varDelta t = -\frac{\varDelta h}{15 \cos L \sin Z},$$

which is a minimum, when $Z = \pm 90°$, and incalculable when $Z = 0$ or $180°$.

† From (118) we find

$$\varDelta t = -\frac{\varDelta L}{15 \cos L \tan Z},$$

which is o, when $Z = \pm 90°$, and also incalculable when $Z = 0$ or $180°$.

forward to the time of observation from a preceding, or carried back from a subsequent determination. It may be very largely in error, especially in uncertain currents, or after running several days without observations.

A small error may also result from the assumption that the mean of the instrumental readings corresponds to the mean of the noted times. The reduction of the mean of the altitudes to the mean of the times can be found,* but it can be avoided by limiting the series of observations, which are combined together, to so brief a period, that the error becomes insensible; or, when the body is near the meridian in azimuth, by reducing each observation by itself. This last case, however, should be avoided in this problem.

198. At sea, it is usual to reduce longitudes obtained from day observations to noon by allowing for the run of the ship in the interval, and for currents when known. Those from night observations are recorded for the time of observation; or reduced to the commencement or end of the watch.

199. 2*d Method.* Altitudes in the forenoon and in the afternoon, or on different sides of the meridian, are preferable to single altitudes for finding the local time, for the reasons already stated in Article 174. The longitudes can be found from each set separately and then combined.

At sea the longitudes derived from each can be reduced to noon, and the mean of the two taken as the true longitude; or, if the difference can be regarded as due to currents, the longitude at noon can be found by interpolating for the elapsed time. It is desirable that the observations should be made at nearly equal intervals from noon.

Longitudes by A. M. and P. M. observations are enjoined

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* Chauvenet's Astronomy, Vol. I., p. 214.

in the directions of the Navy Department whenever practicable.

Examples. (Prob. 51.)

1. At sea, May 17, $9^h\ 45^m$ A. M.; 24° 50′ N., 82° 18′ W. by reckoning from preceding noon;

T. by Watch $9^h\ 30^m\ 15^s$; obs'd altitude of $\underline{\odot}$ 58° 17′; Chro. — Watch + $5^h\ 12^m\ 26^s$; Chro. cor. + $25^m\ 15^s$; Index cor. of sextant + 3′ 20″; height of eye 18 feet; required the longitude.

	h m s		⊙'s dec. ° ′ ″	″	Eq. of t. m s	s
T. by W.	9 30 15					
C—W	+5 12 26		+19 23 40	+33.5	—3 51.1	+0.067
Chro. cor.	+25 15		+1 45	100.5	+.2	.20
G. m. t. May 17	3 7 56	= 3^d.13	+19 25 25	4.4	3 50.9	1
—Eq. of t.	+3 51	$\underline{\odot}$	58 17	In. cor. + 3′ 20″	dip. —4′ 3″	
G. ap. t. May 17	3 11 47		+14 36	S.diam. +15 50	ref.&p. —31	
		$h=$	58 31 36			
		$L=$	24 50	l. sec	0.04214	
		$p=$	70 34 35	l. cosec	0.02545	
		$2s=$	153 56 11			
		$s=$	76 58 6	l. cos	9.35315	
		$s-h=$	18 26 30	l. sin	9.50015	
L. ap. t. May 16	21 45 46				18.92089	
Long.	+5 26 1	or	81° 30′ W.	l. sin $\frac{1}{2}t$.	9.46045	

May 17, noon, lat. by mer. alt. of ⊙, 25° 8′ N.; run of the ship from $9\frac{3}{4}^h$ A. M., E. N. E. (true) 18 miles.

For E. N. E. 18′, $l = 6'.9$ N., $p = 16'.6$ E., $D = 18'.4$ E.

At the time of the A. M. observations, then, the latitude carried back from noon was 25° 1′ N. Using this in the computation of the time, we find the *L. ap. t.* May 16, $21^h\ 45^m\ 29^s$, and the long 81° $29\frac{1}{2}'$ W. Applying $D = 18'.4$ E., we have for the longitude,

May 17, noon, 81° 11′ W., from observations at 9.45 A. M.

By P. M. observations, and reduced to noon, the longitude was found to be,

May 17, noon, 80° 44′ W. from observations at 3.45 P. M.

As the position is in the Gulf Stream, where there is a strong easterly current, the difference of the two longitudes is attributed to that cause. We take, then, as the longitude at noon,

$$81° \; 11' - \frac{2.2 \times 27'}{6} = 81° \; 1' \text{ W.}$$

2. At sea, 1865, Sept. 5, 4½ A. M., lat. 20° 16′ S., long. 74° 20′ W. T. by Chro. $10^h\ 36^m\ 25^s$; Chro. cor. (G. m. t.) $-1^h\ 16^m\ 10^s$; obs'd alt. ☽, 20° 16′ 0″, W. of meridian; Index. cor. + 2′ 20″; height of eye 15 feet; required the longitude.

T. by Chro. 12^h	$+10^h\ 36^m\ 25^s$		☽'s *R. A.*		☽'s *dec.*	
Chro. cor.	−1 16 10		$22^h\ 40^m\ 41.9^s$	$+2.370^s$	−4° 39′ 5″	+11.62″
G. m. t. Sept. 4	21 20 15		+48.0	47.4	+3 55	232
S_0	+10 54 28.3		22 41 29.9	.6	−4 35 10	3
Red. for G. m. t.	+3 30.3	☽	20° 16′ 0″	In. cor.	+2′ 20″	H. par. 60′ 39″
G. sid. t.	8 18 13.6		+15 14	Dip	−3 49	
☽'s *R. A.*	22 41 29.9	$h'=$	20 31 4	S. diam.	+16 33	+10″
☽'s G. h. ang.	+9 36 44		+54 19	Par. and ref.		
		$h=$	20 25 23			
		$L=$	20 16	l. sec	0.02776	
		$p=$	85 24 50	l. cosec	0.00139	
		$2s=$	126 6 13			
		$s=$	63 3 7	l. cos	9.65627	
		$s-h=$	42 37 44	l. sin	9.83074	
					19.51616	
☽'s L. h. ang.	+4 39 37			l. sin ½	9.75808	
Long.	+4 57 7	or	74° 17′ W.			

NOTE.—The examples under Problem 30 can be adapted to this by regarding the chronometer correction given, instead of the longitude.

200. 3*d Method.* (Littrow's. By *double* altitudes of the same body.)

When two altitudes of a body have been observed, and the times noted by the chronometer or watch, the hour-angles and local times can be found from each separately; and thence the longitude for each. But we may also combine them and find the hour-angle for the middle instant between them.

PROBLEM 52. *From two altitudes of a heavenly body, supposing the declination to be the same for both, to find the mean of the two hour-angles, the latitude of the place and the Greenwich time being given.*

Solution. Take the mean of the two noted times and reduce it to Greenwich mean time; and find for it the declination of the body.

Reduce the observed altitudes to true altitudes.

Let h and h' be the two altitudes,
T and T', the corresponding hour-angles;

then we have, by (116),

$$\sin h = \sin L \sin d + \cos L \cos d \cos T,$$
$$\sin h' = \sin L \sin d + \cos L \cos d \cos T';$$

and by subtracting the first from the second,

$$\sin h' - \sin h = \cos L \cos d\,(\cos T' - \cos T).$$

By Pl. Trig. (106) and (108), this reduces to

$$\sin \tfrac{1}{2}(h'-h) \cos \tfrac{1}{2}(h'+h) = -\cos L \cos d \sin \tfrac{1}{2}(T'+T) \sin \tfrac{1}{2}(T'-T);$$

whence

$$\sin \tfrac{1}{2}(T'+T) = -\frac{\sin \tfrac{1}{2}(h'-h) \cos \tfrac{1}{2}(h'+h)}{\sin \tfrac{1}{2}(T'-T) \cos L \cos d}.$$

Put $H_0 = \tfrac{1}{2}(h'+h)$, the mean of the two altitudes,
$T_0 = \tfrac{1}{2}(T'+T)$, the middle hour-angle,
$t = (T'-T)$, the difference of the two hour-angles;

and we have

$$\sin T_0 = \frac{\sin \tfrac{1}{2}(h-h')}{\sin \tfrac{1}{2} t} \cos H_0 \sec L \sec d. \qquad (146)$$

t, for the sun is the elapsed *apparent* time; for a star, the elapsed *sidereal* time; and for the moon or a planet, the elapsed sidereal time — the increase of right ascension in the interval; and can be found from the difference of the two chronometer times.

Then, by (137), T_0 can be found, and, as any other local hour-angle, subtracted from the corresponding Greenwich hour-angle, which in this case is to be derived from the mean of the noted times.

T_0 is + or − according as the second altitude is less or greater than the first; so that it is on the same side of the meridian as the body at the time of its less altitude.

If $(h-h')$ is very small (146) becomes approximately

$$\sin T_0 = \frac{\frac{1}{2}(h-h') \sin 1'' \cos H_0}{\sin \frac{1}{2} t \cos L \cos d}; \tag{147}$$

If T_0 is very small,

$$T_0 = \frac{\sin \frac{1}{2}(h-h') \cos H_0}{15 \sin 1'' \sin \frac{1}{2} t \cos L \cos d}; \tag{148}$$

and if both are small,

$$T_0 = \frac{\frac{1}{2}(h-h') \cos H_0}{15 \sin \frac{1}{2} t \cos L \cos d}. \tag{149}$$

201. To estimate the effect of small errors in the data of the problem, let

$\Delta\,(h-h')$ be a small increment of the difference of the two altitudes,
$\Delta\,H_0$, of the mean of the altitudes,
$\Delta\,L$, of the assumed latitude,
$\Delta\,t$, of the elapsed time, and
$\Delta\,T_0$, the corresponding change of the middle hour-angle; the last two expressed in *seconds* of time, the rest in *minutes* of arc:

then, we have the formula,*

$$\varDelta T_0 = \left[\frac{2\,\varDelta\,(h-h')}{\tan\frac{1}{2}\,(h-h')}, -\frac{\varDelta\,t}{2\tan\frac{1}{2}\,t}, -\frac{4\,\varDelta\,H_0}{\cot\,H_0}, +\frac{4\,\varDelta\,L}{\cot\,L}\right]\tan\,T_0; \quad (150)$$

in which each term may be computed separately for any supposed value of its numerator. The possible error in any case, on the suppositions made, would be the numerical sum of the several terms.

As $h-h'$, however, is the change of altitude in the interval t, we may attribute all the error to $h-h'$, and regard t as exact; or we may attribute all the error to t and regard $h-h'$ as exact: so that one only of the first two terms in the second member of (150) is needed.

When $h-h'$ is so small that we may put $\cos\frac{1}{2}\,(h-h')=1$, we may use instead of (150)

$$\varDelta\,T_0 = \frac{2\,\varDelta\,(h-h').\,\cos\,H_0}{\sin\frac{1}{2}\,t\,\cos\,L\,\cos\,d\,\cos\,T_0}, \quad (151)$$

which is preferable to (150) and requires for computation four of the same logarithms as (147).

The effects of errors are evidently least when $T_0=0$; that is, in the case of equal altitudes each side of the meridian. They increase rapidly as T_0 increases; so that the method is especially adapted for altitudes nearly equal on both sides of the meridian, or for circum-meridian altitudes.

But in the latter case, especially, a high altitude is necessary; for from (151) it appears that the effect of error in

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* Changing (146) into a logarithmic form, we have,

$$\text{l.}\sin\,T_0 = \text{l.}\sin\tfrac{1}{2}\,(h-h') - \text{l.}\sin\tfrac{1}{2}\,t + \text{l.}\cos\,H_0 - \text{l.}\cos\,L - \text{l.}\cos\,d;$$

differentiating each term except the last, and reducing, we obtain,

$$\frac{\mathrm{d}\,T_0}{\tan\,T_0} = \frac{\mathrm{d}\,(h-h')}{2\tan\frac{1}{2}\,(h-h')} - \frac{\mathrm{d}\,t}{2\tan\frac{1}{2}\,t} - \frac{\mathrm{d}\,H_0}{\cot\,H_0} + \frac{\mathrm{d}\,L}{\cot\,L}.$$

This is reduced to (150) by multiplying $\varDelta\,(h-h')$, $\varDelta\,H_0$ and $\varDelta\,L$ by 4, to reduce minutes of arc to seconds of time.

the difference of altitudes, $h-h'$, is least either when H_0 is very near 90°, or t near 12^h; so that, if t is small, H_0 should be quite large.

202. The method presents no special advantages for observations on shore, except in the case of two nearly equal altitudes of a fixed star on opposite sides of the meridian. In the case of the sun and planets, it is necessary to take the change of declination into consideration, to obtain precise results.

The special case for which the method provides is at sea, within the tropics, when the sun passes the meridian at a high altitude. In that case, when by reason of clouds observations near noon only can be made, or it is desired to obtain the longitude as near noon as practicable, let a pair of altitudes, or several pairs, be measured and the times noted with all the precision practicable. The altitudes should be reduced to true altitudes, and one of each pair for the run of the ship in the interval* by the method given in Prob. 58, and in Bowd., p. 183. From each pair the middle apparent time can be found by (146), and the mean of these times subtracted from the mean of the Greenwich apparent times for the longitude.

If the altitude changes uniformly with the time, or nearly so, the mean of several altitudes observed in quick succession can be taken for a single altitude.

If the observations have been made with care, the errors of instrument, refraction, and dip will affect the two altitudes of each pair nearly alike; and if the reduction for the run of the ship is carefully made, the difference of altitudes in comparison with the difference of times will be nearly exact.

203. This method was proposed by M. Littrow, Director

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* This may be avoided, if the course of the ship is at right angles to the bearing of the sun.

of the Vienna Observatory, and has been successfully used by Admiral Wüllerstorf, of the Austrian navy, in 1857 and 1858. It is highly commended by M. Faye in a full discussion of it in the Comptes Rendus of the French Academy, March 7, 1864. It should be used cautiously, and the errors to which the result is liable in any case carefully estimated.

Altitudes greater than 80° and an interval of more than half an hour are recommended, but an intelligent navigator can readily determine by (150), when he can safely depart from these limits. This will be especially the case when the altitudes are on both sides of the meridian.

Examples.

1. 1865, May 16, $11\frac{1}{2}$ A. M., in lat. 25° 15′ N., long. 56° 20′ W., by account; the ship running N. E. (true) 8 knots an hour.

T. by Chro., $2^h\ 32^m\ 23^s$	☉'s true alt., 81° 1′ 0″,
" " " 2 53 11	" " " 83 40 30;

Chronometer correction on G. mean time $+\ 40^m\ 51^s$; required the longitude.

The distance sailed in the interval is 2′.8. The sun's azimuth at the 1st observation is found to be N. 131° E., which differs 86° from the course. The reduction of the 1st altitude to the place of the 2d is (Prob. 58)

$$2'.8 \times \cos 86° = +\ 0'.2 = +\ 12''.$$

	h m s	☉'s *dec.*		*Eq. of t.*
1st chro. t.	2 32 23	° ′ ″	″	m s s
2d " "	2 53 11	+19 10 4	+34.3	−3 52.41+0.05
Elapsed chro. t.	$t =$ 20 48	+1 57	103.	+.17
Mid. " "	2 42 47	+19 12 1	14.	−3 52.2
Chro. cor.	+ 40 51		° ′ ″	
G. m. t. May 16	3 23 38	$h =$	81 1 12	
—Eq. of t.	+ 3 52	$h' =$	83 40 30	
G. ap. t.	3 27 30	$\frac{1}{2}(h-h') =$	− 1 19 39	l. sin 8.3649 n
L. ap. t.	23 41 44	$H_0 =$	82 21	l. cos 9.1242
Long. at 2d obs.	+3 45 46	$L =$	25 15	l. sec 0.0436
	56° 26′.2 W.	$d =$	19 12	l. sec 0.0249
Red. for $0^h.1$	0.6 E.	$t =$	$0^h\ 20^m\ 48^s$	l. cosec $\frac{1}{2}t$ 1.3433
Long. at noon	56 27 W.	$T_0 =$	−0 18 16	l. sin 8.9009 n

By (150), if $\Delta\,(h-h')=+1'$, $\Delta\,T_0=+6^s.9$; if $\Delta\,H_0=+1'$, $\Delta\,T_0=+2^s.4$; if $\Delta\,L=+10'$, $\Delta\,T_0=+1^s.5$.

2. At sea, 1865, June 29, lat. at noon by mer. alt. of ☉, 33° 25′ N., long. by account 147° 10′ E.;
near 11 A.M. T. by Chro., $1^h\,55^m\,54^s$ }
" 1 P.M. " " " 3 45 0 } ☉'s true alt. 74° 21′ 30″;
Chro. cor. on G. m. t. $-36^m\,28^s$; the ship run
from 1st observation, to noon, N. 3 pts. W. 9′.9 }
" noon to 2d observation, N. 2 " W. 7 .2; }
required the longitude at noon.

N. 3 W.	9′.9	8′.2 N.	5′.5 W.	$\Delta\,\lambda = 6'.6$ W.
N. 2 W.	7 .2,	6 .6	2 .8	3 .4
N. 30° W.	17 .0	14 .8	8 .3	10 .0

The sun's azimuth was found to be N. 127° E. at the 1st observation; N. 127° W. at the 2d observation.

The difference of N. 30° W. and N. 127° E. is 157°;

the difference of S. 30° E. and N. 127° W. is 83°.

It will be better, therefore, to reduce the second altitude to the position of the first. By Prob. 58, (or Bowd., p. 183,) this reduction is $17'.0 \times \cos 83° = +\,2'.1$. The latitude at the time of the 1st observation was 33° 16′.8 N.

	h m s			
A. M. chro. t.	13 55 54	*☉'s dec.*	*Eq. of t.*	
P. M. " "	15 45 0	+23° 16′ 59″—7″.3	$+2^m\,55^s.6+0^s.507$	
Elapsed " " $t=$	1 49 6	—1 45	+7 .2	7 .10
Mid. " "	14 50 27	+23 15 14	+3 2 .8	.11
Chro. cor.	— 36 28	h = 74° 21′ 30″	l. cosec $\frac{1}{2}t$	0.627
Mid. G. m. t. June 28	14 13 59	h' = 74 23 36	log $\frac{1}{15}$	8.824
—Eq. of t.	— 3 3	$\frac{1}{2}(h-h')$ = —1 3	log	1.799 n
Mid. G. ap. t.	14 10 56	H_0 = 74 22 33	l. cos	9.430
" L. ap. t.	23 59 54	L = 33 17	l. sec	0.078
Long. at 1st obs.	—9 48 58	d = 23 15	l. sec	0.037
	147° 14′.5 E.	T_0 = —6^s.3	log	0.795 n
Red. to noon	6 .6 W.			
Long. at noon	147 7 .9 E.			

By (151) if $\Delta\ (h—h') = 1'$, $\Delta\ T_0 = 3^s.1$. It would require a change of $1\frac{1}{2}°$ in L, or of $2\frac{1}{2}°$ in H_0, for either to change T_0 one second of time. The accuracy of the result, therefore, depends upon the accuracy with which the difference of altitudes has been found; that is, in this case, mainly upon the course and distance made good.

204. *4th method.* (By *equal* altitudes.) Let equal altitudes of a heavenly body be observed east and west of the meridian (Art. 175) and the times noted as in other observations; and the mean of the watch-times in each set, if a watch is used, reduced to chronometer time. If both sets have been observed at the same place, and the declination of the body has not changed, the mean of the two times will be the chronometer time of its meridian transit.

If the declination has changed in the interval, as is ordinarily the case with the sun, moon, or a planet, the correction for such change, found by the methods of Problem 51, should be applied.

Applying then the chronometer correction, we have the corresponding Greenwich time, which will be mean or sidereal as the time to which the chronometer is regulated.

Finding from this, by the method in Art. 189, the *Greenwich hour-angle* of the body (which in the case of the sun is the Greenwich apparent time), we have the longitude, if the first observation was *east* of the meridian, as the corresponding *local hour-angle* is then 0. But if the first observation was west of the meridian, the local hour-angle is 12^h and must be subtracted.

This method should be used on shore, when practicable, in preference to either of the preceding.

205. Equal altitudes of the sun can be conveniently used at sea when the sun passes the meridian near the zenith; that is, when its declination and the latitude are nearly the same. Altitudes very near noon are then available for finding the time (Art. 196), and equal altitudes can be observed

with only a short interval. In the example of Art. 196, an interval of eight minutes would have been sufficient.

If the ship does not change her position in the interval, the middle time corresponds to apparent noon; as the change of declination may be neglected, unless the interval between the observations is so great as to require it.

206. If the longitude only has changed, the middle time corresponds to apparent noon at the middle meridian, and will give the longitude of that meridian. This will be the longitude at noon, if the speed of the ship has been uniform. But if it has not, subtracting half the change of longitude, when the true course is *west*, or adding it when the course is *east*, will give the longitude of the place where the first altitude was observed. This can then be reduced to noon by allowing for the run of the ship.

If the change of longitude is west, the sun arrives at the corresponding altitude of the afternoon later than it would do if observed at the same place as in the forenoon; if the change is east, it arrives earlier; and the difference is the time of the sun's passing from the one meridian to the other, that is, the difference of longitude expressed in time.

If, then, $2\,t$ is the elapsed apparent time,

$\varDelta\lambda$, the change of longitude (+ when west), the hour-angle of the sun at each observation is $t - \frac{1}{2}\varDelta\lambda$; and (137) becomes

$$\varDelta T_0 = -\frac{\varDelta_h d.\ t \tan L}{15 \sin (t - \frac{1}{2}\varDelta\lambda)} + \frac{\varDelta_h d.\ t \tan d}{15 \tan (t - \frac{1}{2}\varDelta\lambda)}, \qquad (152)$$

But even when the elapsed time is so great that it is thought necessary to correct for the change of declination, $\varDelta\lambda$ is never large enough to produce a change of 1^s.

If the latitude only has changed, the middle time requires correction for such a change, which can be deduced in a similar way to that for a change of declination in Prob. 51. But, as in the fundamental formula,

$$\sin h = \sin L \sin d + \cos L \cos d \cos t,$$

L and d enter with the same functions, they are interchangeable. If, then,

$\Delta_h L$ is the hourly change of latitude (+ toward the north and expressed in seconds), and

$\Delta' T_0$, the required correction,

we have from (137) and (139),

$$\Delta' T_0 = -\frac{\Delta_h L.\ t \tan d}{15 \sin t} + \frac{\Delta_h L.\ t \tan L}{15 \tan t} \qquad (153)$$

and $$\Delta' T_0 = A\ \Delta_h L.\ \tan d + B\ \Delta_h L.\ \tan L, \qquad (154)$$

for which Chauvenet's tables can be used.

If both latitude and longitude have changed, for t in the denominators of (153), we may substitute $t - \frac{1}{2} \Delta \lambda$: but this at sea is a needless refinement.

The restriction of this method to a short interval between the observations, depends upon the uncertainty of the run of the ship and consequent imperfect determination of $\Delta_h L$, the mean hourly change of latitude in the interval. If its error is supposed to be $\frac{1}{n} \Delta_h L$, the consequent error in $\Delta' T_0$ is $\frac{1}{n} \Delta' T_0$.

When equal altitudes near noon are practicable, a meridian altitude of the sun can ordinarily be taken for latitude, so that L will be sufficiently exact. Moreover, the latitude and longitude are both found for noon.

Examples.

1. At sea, 1865, March 17, noon, lat. by mer. alt. of the sun 3° 16′ S., long. by account 84° 58′ W.; equal altitudes of the sun were observed at $5^h\ 34^m\ 18^s$ and $6^h\ 3^m\ 24^s$ G. mean time; the ship running S. S. E. (true) 10 knots an hour; required the longitude.

For S. S. E., 10′, $\Delta_h L = -9'.2$, $\Delta_h \lambda = -3'.8$

	h m s				
1st G. m. t. March 17	5 34 18	⊙'s *dec.*		*Eq'n of t.*	
2d G. m. t.	6 3 24	−1° 13′ 10″	+59″.25	$+8^m\ 27^s.5$	$-0^s.736$
Elapsed time	0 29 6	+5 44	296.	−4 .3	3 .7
Mid. G. m. t. March 17	5 48 51	−1 7 26	48.	+8 23 .2	.6
—Eq. of t.	−8 23	$\Delta_h L = -552''$	log 2.742 *n*	log	2.742 *n*
Mid. G. ap. t. March 17	5 40 28	$L = -3° 16'$			l. tan 8.756 *n*
Red. for ΔL	+5	$d = -1$ 7	l. tan 8.290 *n*		
G. ap. t. of noon	5 40 33		log A 9.406 *n*	log B 9.405	
or long.	85° 8′ W.	$-2^s.7$	log 0.438 *n*		
		+8 .0		log	0.903

In this example the sun's azimuth was 120°, and in 1^m the altitude changed 13′. An inequality of 30″ in the altitudes would therefore affect the result only $\frac{1}{52}$ of 1^m, or $1^s.2$. An error of 1′ in the hourly change of latitude would affect the result $\frac{5^s}{9.2}$, or $0^s.6$.

2. At sea, 1865, June 16, lat. at noon by mer. alt. of ⊙, 22° 50′ N., long. by account 35° 59′ W.; equal altitudes of the sun were observed at $2^h\ 16^m\ 18^s$ and $2^h\ 31^m\ 42^s$ G. mean time; the ship running S. (true) 14′ an hour.

Elapsed time		$0^h\ 15^m\ 24^s$	⊙'s dec. +23° 22′.4
Mid. G. m. t.	June 16	2 24 0	
—Eq. of t.		−22.5	
Mid. G. ap. t.		2 23 37.5	
Red. for ΔL		+2.5	
G. ap. time of noon		2 23 40	
Long.		35° 45′ W.	

The sun's azimuth was 72°; the change of altitude in 1^m was 13′.2, so that an inequality of 1′ in the altitudes would affect the result $\frac{1}{26}$ of 1^m, or $2^s.3$. An error of 1′ in $\Delta_h L$ would affect the result $\frac{2^s.5}{14}$, or $0^s.2$.

3. At sea, 1865, June 29, 0^h; lat. by mer. alt. of ⊙, 33° 25′ N., long. by account 147° 10′ E.;

near 11 A. M., T. by Chro. $1^h\ 55^m\ 54^s$ } obs'd alt. of ☉ 74° 9′ 10″;
" 2 P. M., " " " 3 45 0 }
Chro. cor. on G. m. t. $-36^m\ 28^s$; In. cor. of sex't +0′ 50″; height of eye 18 feet. The ship run
from 11 A. M. to noon N. 3 p'ts W. 11′ }
from noon to 3 P. M. N. 2 " W. 8′ }
required the longitude at noon.

For N. 3 W. 11′	$\Delta L = +9'.1$	$\Delta\lambda = +7'.4$
N. 2 W. 8	$\Delta L = +7.4$	$\Delta\lambda = +3.7$
whence	$\Delta_h L = +8.25 = 495''$	

	h m s		
A. M. chro. t. $+12^h$	13 55 54	☉'s *dec.*	*Eq'n of t.*
P. M. chro. t.	15 45 0	+23° 16′ 59″ − 7″.3	$+2^m\ 55^s.6 + 0^s.51$
Elapsed time	1 49 6	−1 45	+7 .2 } 7.14
Mid. chro. t.	14 50 27	+23 15 14	+2 2 .8 } .11
Chro. cor. (G. m. t.)	−36 28		
Mid. G. m. t. June 28	14 13 59	log $\Delta_h L$ 2.695	log $\Delta_h L$ 2.695
−Eq. of t.	−3 3	log A 9.410 n	log B 9.398
Mid. G. ap. t.	14 10 56	l. tan d 9.633	l. tan L 9.819
Red. for ΔL	+27	log 1.738 n	log 1.912
G. ap. t. of noon	14 11 23	$-54^s.7 + 81^s.7 = +27^s$	
Middle long.	−9 48 37 or 147° 9′.2 E.		
Red. to noon	1 .8 W.		
Long. at noon	147 7 .4 E.		

The sun's azimuth was 127°; for $\Delta t = 1^m$, $\Delta h = 10''$, and an inequality of 1′ in the altitudes will effect the result $\frac{1}{20}$ of 1^m, or 3^s. An error of 1′ in $\Delta_h L$ will affect the result $\frac{27^s}{8.25}$, or $3^s.3$.

These observations reduced as single altitudes, give, as the longitude at noon, 147° 7′.8 E.; reduced by Littrow's method (Ex. 2, p. 202), 147° 7′.9 E.

207. *5th Method.* (By *transits.*)

Observe the transits of the sun or a star across the threads of a well-adjusted transit instrument, noting the times. Re-

duce the mean of the noted times for semi-diameter and errors of the instrument as in Art. 184; and thence find the Greenwich hour-angle of the body in the way described in Art. 189. This will be the longitude, if the upper culmination has been observed, as the local hour-angle is 0. If the lower culmination has been observed, the local hour-angle is 12^h.

This method can be used only on shore.

Example.

1865, May 17, $17^h\ 16^m\ 20^s.5$ G. mean time, the meridian transit of *a* Bootis (*Arcturus*) was observed; required the longitude of the place of observation.

G. mean time	May 17	17^h	16^m	$20^s.5$	
S_0		3	40	47.20	
Red. for G. m. t,			+2	50.24	
G. sid. t.		20	59	57.94	
∗'s R. A.		14	9	32.81	
∗'s H. angle or Long.		+6	50	25.1	or 102° 36′ 17″ W.

LONGITUDE.—LUNAR DISTANCES.

208. Problem 53. *To find the longitude by the distance of the moon from some other celestial object.*

Solution. If we have given the local mean time and the *true* distance of the moon from some celestial object as seen from the centre of the earth, we may find, by interpolating the Nautical Almanac lunar distances (Prob. 28), the Greenwich mean time corresponding to this distance. The difference of this from the local time is the longitude.

The local time may be found for the instant of observation, either from an altitude of a celestial object observed at the same time, or by a chronometer regulated to the local time.

At sea the correction of the chronometer on local time can

be found from altitudes observed near the time of measuring the lunar distance, and reduced for the change of longitude in the interval by the formula (Art. 167),

$$c' = c + \Delta\,\lambda,$$

$\Delta\,\lambda$ being in time and + when the change is west.

In practice, the *apparent* distance of the moon's bright limb from the sun or a star is observed, and the *true* distance derived by calculation, as in the next problem.

209. PROBLEM 54. *Given the apparent distance of the moon's bright limb from a star, the centre of a planet, or the sun's nearest limb, to find the true distance of the moon's centre from the star, or the centre of the planet or the sun.*

Solution. It is necessary that the altitudes of the two bodies should be known, either directly from observations at the same time, or from observations before and after, and interpolated to the time of observation (Bowd., p. 246); or computed from the local time (Prob. 38), (Bowd., pp. 247, &c.).

The Greenwich time is also supposed to be known approximately, either from the local time and approximate longitude, or, as is preferable, from the time noted by a Greenwich chronometer.

A complete record of the observations will include the approximate latitude and longitude of the place, the local time and chronometer correction, the index corrections of the instruments used, the height of the barometer and thermometer, and at sea, the height of the eye above the water, as well as the noted times of observation and the observed distances and altitudes. Several observations may be made at brief intervals, and the means taken.

210. The preparation of the data embraces:

1. Finding the Greenwich mean time approximately from the chronometer time, or from the local time.

2. Taking from the Almanac for this time the semi-diame-

ter and horizontal parallax of the moon, and of the other body* when they are of sensible magnitude; adding to the moon's semi-diameter its augmentation. (Art. 60.)

At low altitudes the contractions produced by refractions should be subtracted from the semi-diameters of the sun and moon. Formulas for finding these are given in Art. 213.

When the spheroidal form of the earth is taken into consideration, to the moon's equatorial horizontal parallax (Art. 57), as taken from the Almanac, should be added the augmentation to reduce to the latitude of the place, which is found in Tab. III. of Chauvenet's Method. The declinations of the two bodies to the nearest degree are required from the Almanac for this purpose.

3. Applying to the observed distance the index correction of the instrument, and, when the sun is used, adding the moon's augmented semi-diameter and the sun's semi-diameter; when a planet or star is used, adding the moon's augmented semi-diameter if its nearest limb is observed, but subtracting it if the farthest limb is observed.

4. Applying to the observed altitude of each body the index correction, dip, and semi-diameter (when necessary), so as to find the *apparent* altitude of its centre. If the true altitude is computed, the parallax must be subtracted and the refraction added.

In the following direct method it is necessary also to find the *true* altitudes.

211. To find the *true* distance,

let $D =$ the *apparent* distance of the centres,

$D' =$ the approximate *true* distance,

$\left.\begin{array}{l} h = \text{the } apparent \text{ altitude} \\ h' = \text{the } true \text{ altitude} \end{array}\right\}$ of ☽'s centre,

$\left.\begin{array}{l} H = \text{the } apparent \text{ altitude} \\ H' = \text{the } true \text{ altitude} \end{array}\right\}$ of ⊙'s centre, planet, or star.

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* The sun's horizontal parallax may be taken as $8''.5$.

In Fig. 35, let m and S be the apparent places of the moon and other body; m' and S′, their true places.

The true and apparent places of each are on the same vertical circle, Z m, Z S respectively, since they differ only by refraction and parallax, which act only in vertical circles, except so far as a small term of the moon's parallax is concerned, which will be subsequently considered.

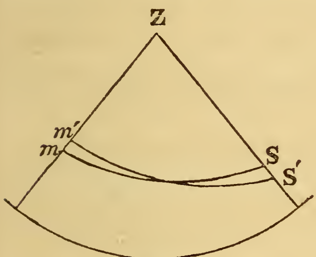

Fig. 35.

Then m S $= D$, the apparent distance;

m' S′$=D'$, the true distance;

and in the triangle m Z S,

$$\left.\begin{array}{l} m\,\mathrm{S} = D \\ \mathrm{Z}\,m = 90° - h \\ \mathrm{Z\,S} = 90° - H \end{array}\right\} \text{being given,}$$

to find the angle Z, we have by Sph. Trig. (32),*

$$\cos^2 \tfrac{1}{2}\,\mathrm{Z} = \frac{\cos \frac{1}{2}(h+H+D) \cos \frac{1}{2}(h+H-D)}{\cos h \cos H}.$$

Then in the triangle m' Z S′,

$$\mathrm{Z}\,m' = 90° - h' \qquad \text{and} \qquad \mathrm{Z\,S'} = 90° - H'$$

being given, m' S′ may be found by Sph. Trig. (17),†

$$\sin^2 \tfrac{1}{2} D' = \cos^2 \tfrac{1}{2}(h'+H') - \cos h' \cos H' \cos^2 \tfrac{1}{2}\,\mathrm{Z},$$

or by substituting the value of $\cos^2 \frac{1}{2}$ Z, and putting

$$s = \tfrac{1}{2}(h+H+D), \tag{155}$$

$$\sin^2 \tfrac{1}{2} D' = \cos^2 \tfrac{1}{2}(h'+H') - \frac{\cos h' \cos H'}{\cos h \cos H} \cos s \cos (s-D).$$

To adapt this for logarithmic computation put

$$\sin^2 \tfrac{1}{2}\,m = \frac{\cos h' \cos H'}{\cos h \cos H} \cos s \cos (s-D), \tag{156}$$

---

* $$\cos^2 \tfrac{1}{2} A = \frac{\sin \frac{1}{2}(a+b+c) \sin \frac{1}{2}(b+c-a)}{\sin b \sin c}.$$

† $\sin^2 \frac{1}{2} a = \sin^2 \frac{1}{2}(b+c) - \sin b \sin c \cos^2 \frac{1}{2} A.$

then

$$\sin^2 \tfrac{1}{2} D' = \cos^2 \tfrac{1}{2} (h' + H') - \sin^2 \tfrac{1}{2} m,$$

which by Pl. Trig. (134), becomes

$$\sin^2 \tfrac{1}{2} D' = \cos \tfrac{1}{2} (h' + H' + m) \cos \tfrac{1}{2} (h' + H' - m),$$

or, if we put

$$s' = \tfrac{1}{2} (h' + H' + m), \tag{157}$$

we have

$$\sin \tfrac{1}{2} D' = \sqrt{[\cos s' \cos (s' - m)]}. \tag{158}$$

The solution is effected by formulas (155), (156), (157), and (158).

This is only one of several direct trigonometric solutions. It is easily remembered, involving only cosines in the second members. But in all such methods 7-place logarithms are required for the computations.

212. If the moon's augmented parallax has been used, the distance obtained, D', is not the *true* distance as seen from the centre of the earth, but from the point C' (Fig. 36), where the vertical line of the place intersects the earth's axis.

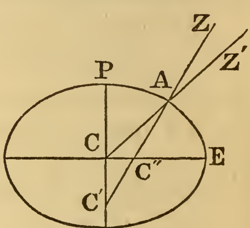

Fig. 36.

A reduction to the centre, C, is still required, for which we have the formula—*

$$\varDelta D' = A \pi \sin L \left(\frac{\sin \delta_s}{\sin D'} - \frac{\sin \delta_m}{\tan D'} \right), \tag{159}$$

in which

δ_s is the sun's declination,

δ_m, the moon's declination,

π, the moon's equatorial horizontal parallax, whose mean value is $57' \, 30''$,

A, a coëfficient depending on the eccentricity of the terres-

* Chauvenet's Astronomy, Vol. I., p. 399.

trial meridian, the mean value of which, for latitude 45°, is .0066855, or of log A, 7.8251,

$A \sin L$, the distance C C′, with C E = 1.

The mean values of $A\pi = 23''.07$, or $\log A\pi = 1.3630$, may be used, unless great precision is required.

The signs of the declinations and latitude are + when *north*, and $\Delta D'$ is to be added algebraically to D'.

If the augmentation of the parallax has been neglected, the distance has been reduced to a point on the vertical line between C′ and C″ and at a distance from A equal to the equatorial radius C E.

213. To find the corrections needed for the contraction by refraction of the semi-diameters of the sun and moon in the direction in which the distance is measured,

let q = the angle Z S m (Fig. 35), at the sun or star,

Q = the angle Z m S, at the moon,

Δs and $\Delta' s$, the contractions of the sun's semi-diameter respectively in the vertical direction S Z, and in the direction of the distance S m;

ΔS and $\Delta' S$, the contractions of the moon's semi-diameter respectively in the vertical direction m Z, and in the direction of the distance m S.

To find q and Q from the three sides of the triangle Z S m, putting, as in (155),

we have

$$\left.\begin{aligned} s &= \tfrac{1}{2}(h+H+D) \\ \sin \tfrac{1}{2} q &= \sqrt{\left(\frac{\cos s \sin (s-H)}{\sin D \cos h}\right)} \\ \sin \tfrac{1}{2} Q &= \sqrt{\left(\frac{\cos s \sin (s-h)}{\sin D \cos H}\right)} \end{aligned}\right\} \qquad (160)$$

for which it will suffice to use a rough approximation of D, and for the computation logarithms to four places; as q and Q are required only within 30′.

The contractions, Δs and ΔS, of the vertical semi-diame-

ters may each be found from the refraction table, by taking the difference of refractions for the limb and centre.

Then, for the required corrections, we have the formulas,*

$$\varDelta' s = \varDelta\, s \cos^2 q, \qquad \varDelta' S = \varDelta\, S \cos^2 Q. \qquad (161)$$

This contraction for either body is less than 1″, if the altitude is greater than 40°. For a very low altitude, it is best to subtract it from the semi-diameter in the preparation of the data, so that D will be corrected for it. But, unless quite large, it will suffice to compute it subsequently, and subtract it from D' when the nearest limb is used, or add it to D' when the farthest limb is used.

214. Let $\varDelta\, D =$ the reduction of the apparent distance to the true, or $D' = D + \varDelta\, D$.

A great variety of methods have been given for finding $\varDelta\, D$, requiring 4 or, at the most, 5-place logarithms; but also needing special tables. Four such methods are contained in Bowditch's Navigator. They generally neglect to take into account the spheroidal form of the earth, the correction of refraction for the barometer and thermometer, and the contraction of the semi-diameters of the sun and moon.

These together, at very low altitudes and in extreme cases, may produce an error of 3^m in the calculated Greenwich time, and do actually, in the average of cases, produce errors from 10^s to 1^m.

Prof. Chauvenet has given in the American Ephemeris for 1855,† a new form to the problem, with convenient tables, by which all these corrections are readily introduced. It is but little longer than the other approximative methods, in which they are neglected.

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* Chauvenet's Astronomy, Vol. I., p. 186.

† Reprinted in a pamphlet with his method of equal altitudes.

215. The moon's mean change of longitude is 13°.17640 in a day (Herschel's Ast., p. 222), or 33″ in 1^m of time.

An error, then, of 33″ in the distance will, in the average, produce an error of 1^m in the Greenwich time, or 15′ in the longitude; or an error of 10″ in the distance will produce an error of about 20^s in the Greenwich time, or 5′ in the longitude.

We may, however, readily find the effect of an error of 1″, and thence any number of seconds, in the distance, by taking the number corresponding in a table of common logarithms to the "Prop. Log. of Diff." in the Almanac; for this prop. log. is simply the logarithm of the change of time in seconds for a change of 1″ in the distance. (p. 95.)

216. Errors of observation are diminished by making a number of measurements of the distance. But even with a skilful observer a single set of distances is liable to a possible error of 10″ or even 20″.

Errors of the instrument are diminished by combining results from distances of different magnitudes, especially from those measured on opposite sides of the moon. This cannot usually be done with longitudes at sea, but may be with determinations of the chronometer correction. The error peculiar to the observer, that is, in making the contacts always too close, or always too open, is not eliminated in this way, but will remain as a constant error of his results.

The accuracy of the reductions of the observed to the true distance, depends more upon the precision with which the differences of the apparent and true altitudes — that is, the parallax and refraction — have been introduced, than upon the accuracy of the altitudes themselves.

217. Lunar distances are used at the present day, not so much for finding the longitude, as for finding the Greenwich mean time, with which to compare the chronometer. They may thus serve as *checks* upon it, which in protracted

voyages may be much needed. If the chronometer correction thus determined agrees with that derived from the original correction and rate, the chronometer has run well, and its rate is confirmed; if otherwise, more or less doubt is thrown upon the chronometer, according to the degree of accuracy of the lunar observation itself. If the discordance is not more than 20^s, it is well still to trust the chronometer, as the best observed single set of distances may give a result in error to that extent. If it is large, then by repeated measurements of lunar distances, differing in magnitude, and especially on both sides of the moon, and carefully reduced, the chronometer correction can be found quite satisfactorily. By taking the rate into consideration, observations running through a number of days can be combined.

Example.

At sea, 1855, Sept. 7, about 6^h A. M., in lat. 35° 30′ N., long. 30° W. by account;
Time by chro. $8^h\ 29^m\ 57^s.5$; app. chro. cor. (G.m.t.) $-21^m\ 1^s.5$;
Observed distance of ☉ and ☽ 43° 52′ 30″, index cor. −20″;
Observed altitude of ☽ 49° 31′ 50″, index cor. +1′ 0″;
Observed altitude of ☉ 5° 27′ 10″, index cor. 0″;
Bar. 29.10 inches; ther. 75°; height of eye 20 feet;
Required from these observations the chronometer correction on Greenwich time.

Preparation.

	h m s		′ ″		″
T. by chro. 12^h+	8 29 57.5	☽'s H. par.	54 19.4	☽'s S. diam.	14 50.0
Chro. cor.	−21 1.5	Aug.	+3.6	Aug.	+11.2
G. m. t. Sept. 6	20 8 56	☽'s Aug. H. par.	54 23.0	☽'s Aug. S. diam.	15 1.2

☽	49° 31′ 50″	☉	5° 27′ 10″	☉'s H. par.	8″.5
In. cor.	+1 0	In. cor.	0		
Dip	−4 23	Dip	−4 23	☉'s S. diam.	15′ 55″.1
Aug. S. diam.	+15 1	V. S. diam.	+15 34	V. cont.	−21 .6
				V. S. diam.	15 33 .5

	$H = 49° 43'' 28'$		$h = 5° 38'' 21'$
Ref.	-46	Ref.	$-8\ 12$
Par.	$+35\ 10$	Par.	$+8$
	$H' = 50\ 17\ 52$		$h' = 5\ 30\ 17$

$H = 49° 43'$		Obs'd dist. ☽ ☉	$43° 52' 10''$
$h = 5\ 38$	l. sec 0.002	☽'s Aug. S. diam.	$+15\ \ 1.2$
$D = 44\ 22$	l. cosec 0.155	☉'s S. diam.	$+15\ 55.1$
$2s = 99\ 43$		Cont. $\varDelta's =$	-21.6
$s = 49\ 52$	l. cos 9.809		$D = 44\ 22\ 45$
$s - H = 0\ 9$	l. sin 7.418	*Cont.* ☉*'s S. diam.* (161).	
	17.384	2 log cos q	9.999
$q = 2\ 49$	l. sin ½ 8.692	log 21.6	1.334
		log $\varDelta's$	1.333

Computation of True Distance. (155–158)

° ′ ″	° ′ ″		
$H = 49\ 43\ 28$		l. sec H	0.1894554
$h = 5\ 38\ 21$		l. sec h	0.0021069
$D = 44\ 22\ 45$	$H' = 50\ 17\ 52$	l. cos H'	9.8053633
$2s = 99\ 44\ 34$	$h' = 5\ 30\ 17$	l. cos h'	9.9979925
$s = 49\ 52\ 17$		l. cos s	9.8092266
$s - D = 5\ 29\ 32$	° ′ ″	l. cos $(s-D)$	9.9980017
	$\frac{1}{2}(H'+h' = 27\ 54\ \ 4.5$		19.8021464
	$\frac{1}{2}m = 52\ 46\ 39.4$	l. sin $\frac{1}{2}m$	9.9010732
	$s' = 80\ 40\ 44$		
	$m = 105\ 33\ 19$	l. cos s'	9.2094277
	$m - s' = 24\ 52\ 35$	l. cos $(m-s')$	9.9577114
	° ′ ″		19.1671391
	$\frac{1}{2}D' = 22\ 32\ 23.7$	l. sin	9.5835696
	$D' = 45\ \ 4\ 47$		
	Cor. for Comp. -4		
	$D'' = 45\ \ 4\ 43$	*true distance.*	

Compression. (159)

☉'s dec. $= +6°.3$	l. sin	9.040
D'	l. cosec	0.150
$n = 0.155$	log	9.190
☽'s dec. $= +25°.3$	l. sin	9.631
D'	l. cot	9.999
$n' = .427$	log	9.630
$n - n' = -.272$	log	9.435 n
$A\pi$	log	1.363
L	l. sin	9.764
Comp. $-3''.6$	log	0.562 n

Finding the Greenwich mean time and chronometer correction. (82)

True distance	$D'' = 45°\ \ 4'\ 43''$		
Distance at 18^h.	$D_0 = 46\ \ 3\ 17$	P. L. 0.3433	Diff. +5
	$D'' - D_0 = 58\ 34$	log 3.5458	
	$t = 2^h\ 9^m\ 6^s$	log 3.8891	

G. m. t. of D_0	18^h	0^m	0^s	
Red. for 2d diff.			-2	
G. mean time, Sept. 6	20	9	4	
T. by chro.	20	29	57	
Chro. cor.		-20	53	by lunar.
		-21	1	by previous cor. and rate.
Difference			$+8$	

This example is taken from the pamphlet of Prof. Chauvenet, where it is reduced by his method with far less labor of computation. The true distance by that method is 45° 4′ 45″; by Bowditch's 1st method, in which the small corrections are omitted, it is 45° 5′ 44″, differing very nearly 1′ from the correct value. This would produce an error of 2^m 10^s in the Greenwich time.

218. Other lunar methods for finding the longitude, beside that of lunar distances, are—

1. By *moon culminations*, or observing the meridian transits of the moon and several selected stars near its path, whose right ascensions are considered well determined.

2. By *occultations*, or noting the instant that a star disappears by being eclipsed by the moon, or that it reäppears from behind the moon. The first is called an *immersion*, the second an *emersion*.

3. By *altitudes* of the moon near the prime vertical.

4. By *azimuths* of the moon and stars observed near the meridian.

These methods, except occasionally the second, are available only on shore. They require good instruments, careful observations and determinations of the instrument corrections, and scrupulous exactness in the reductions, especially those which involve the moon's parallax.

By each may be found the moon's right ascension, and thence, by inverse interpolation in the Almanac, the corresponding Greenwich mean time. Subtracting from it the local mean time, which must also be found from good observations, gives the longitude.

219. If corresponding observations are made at two dif-

ferent places, their difference of longitude can be found with much less dependence on the accuracy of the Ephemeris.

When the two local times of the occultation of the same star have been noted, they can each be reduced to the instant of the *geocentric* conjunction of the moon's centre and the star in right ascension ; and the difference of the reduced times will be the longitude.

By the other methods, the change of the right ascension of the moon, in passing from one meridian to the other, may be found. This, divided by the mean change in a unit of time, as 1^h or 1^m, computed from the Ephemeris, will give the difference of longitude in the same unit.

CHAPTER IX.

SUMNER'S METHOD: LATITUDE AND LONGITUDE BY DOUBLE ALTITUDES.

CIRCLES OF EQUAL ALTITUDE.—(SUMNER'S METHOD.)

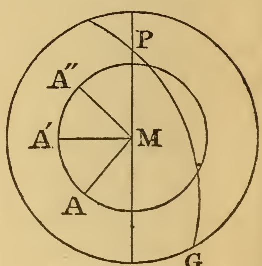

Fig. 37.

219. SUPPOSE that at a given instant the sun, or any other heavenly body, is in the zenith of the place M (Fig. 37), on the earth; and let A A′ A″ be a small circle described from M as a pole. The zenith distance of the body will be the same at all places on this small circle, namely, the arc M A; for if the representation is transferred to the celestial sphere, or projected on the celestial sphere from the centre as the projecting point,

M will be the place of the sun, or other body, and the circle A A′ A″ will pass through the zeniths of all places on the terrestrial circle, and

M A, M A′, &c., will be equal zenith distances.

The altitude of the body will also be the same at all places on the terrestrial circle A A′ A″; hence such a circle is called a *circle of equal altitude.*

It is evident that this circle will be smaller the greater the altitude of the body.

220. The latitude of M is equal to the declination of the body, and its longitude is the Greenwich hour-angle of the

body; which, in the case of the sun, is the Greenwich apparent time, or 24^h—that apparent time, according as the time is less or greater than 12^h. This is evident from the diagram, in which, regarded as on the celestial sphere,

P M is the celestial meridian of the place, whose zenith is M, and its co-latitude; and also the declination circle, and co-declination, of the body M;

and if P G is the celestial meridian of Greenwich, G P M is, at the same time, the longitude of the place, and the Greenwich hour-angle of the body.

If, then, the Greenwich time is known, the position of M may be found and marked on an artificial globe.

221. If, moreover, the altitude of the body is measured, and a small circle is described on the globe about M as a pole, with the complement of the altitude as the polar radius, the position of the observer will be at some point of this circle. His position, then, is just as well determined as if he knew his latitude alone, or his longitude alone; since a knowledge of only one of these elements simply determines his position to be on a particular circle, without fixing upon any point of that circle.

As, however, he may be presumed to know his latitude and longitude approximately, he will know that his position is within a limited portion of this circle. Such portion only he need consider. It is commonly called a *line of position.**

222. The direction of this line at any point is at right angles with the direction of the body; for the polar radius M A is perpendicular to the circle A A′ A″ at A, A′, A″, and every other point of the circle.

223. Artificial globes are constructed on so small a scale that the projection of a circle of equal altitude on a globe

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* Inappropriately termed *a line of bearing.*

would give only a rough determination. But the projection of a limited portion may be made upon a chart by finding as many points of the curve as may be necessary, and, having plotted them upon the chart, tracing the curve through them. The portion required is usually so limited that, when the altitude of the body is not very great, it may be regarded as a straight line; and hence two points suffice. With high altitudes, three points, or if the body is very near the zenith, four may be necessary, and even the entire circle may be required.

224. PROBLEM 55. *From an altitude of a heavenly body to find the line of position of the observer, the Greenwich time of the observation being known.*

Solution. From the given altitude, and assumed latitudes L_1, L_2, L_3, &c., differing but little from the supposed latitude, find the corresponding local times (Prob. 43), and thence, by the Greenwich time, the longitudes λ_1, λ_2, λ_3, &c. Thus we shall have the several points, whose positions are conveniently designated as (L_1, λ_1,), (L_2, λ_2,), (L_3, λ_3,), &c.

It facilitates the computation to assume latitudes differing 10′ or 20′, as the ½ sums and remainders differ 5′ or 10′, and only one of each need be written.

Or, from the Greenwich time and assumed longitudes, λ_1, λ_2, λ_3, &c., find the corresponding local times (Art. 77), and thence the hour-angles of the body (Probs. 34, 35). With these and the observed altitude, find the corresponding latitudes , L_1, L_2, L_3, &c. (Prob. 46).

This is more convenient than the preceding method, when the body is near the meridian.

In either mode the computation may be arranged so that the like quantities in the several sets shall be in the same line, and taken out at the same opening of the tables.

The several points may then be plotted on a chart, each by its latitude and longitude, and a line traced through

them, which will be the required *line of position.* Two points connected by a straight line are sufficient, unless the altitude is very great, or the points widely distant.

Thus in (Fig. 38), let A and B be two such points plotted respectively on the parallels of latitude L_1, L_2, and each in its proper longitude; A B is the *line of position*, and the place of observation is at some point of A B, or A B produced.

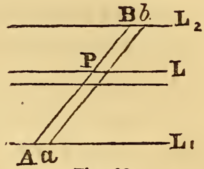

Fig. 38.

This is *all* which can be determined from an observed altitude, unless either the latitude, or the longitude, is definitely known. And as these are both uncertain at sea, except at the time when found directly by observation, the position of the ship found from a single altitude, or set of altitudes, is a *line*, of greater or less extent as the latitude, or the longitude, is more or less accurately known.

In uncertain currents, or when no observations have been had for several days, the extent of this line may be very great. Yet, if it is parallel to the coast, it assures the navigator of his distance from land; if directed toward some point of the coast, it gives the bearing of that point.

225. If there is uncertainty in the altitude, for instance of 3′, the line of position having been computed and plotted, parallels to it on each side may be drawn at the distance of 3′.

So, also, if there is uncertainty in the Greenwich time, parallels may be drawn at a distance in *longitude* equal to the amount of uncertainty.

In either case, the position of the ship is within the inclosed belt.

In Fig. 38, *a b* is such a parallel to the line of position A B, its perpendicular distance from it measuring a difference of altitude; the distance A *a* on a parallel of latitude measuring a difference of longitude.

226. Since the line of position is at right angles with the direction of the body (Art. 222), the nearer the body is to the meridian in azimuth, the more nearly the line of position coïncides with a parallel of latitude; and thus a position of the body near the meridian is favorable for finding the latitude from an observed altitude, and not the longitude.

So also, the nearer the body is to the prime vertical, the more nearly the line of position coïncides with a meridian, and the less does any error in the assumed latitude affect the longitude computed from an observed altitude. So that, if the body is on the prime vertical, a very large error in the assumed latitude will not sensibly affect the result. Such a position of the body is, then, the most favorable for finding the longitude from an observed altitude.

These conclusions have been previously stated, drawn from analytical considerations.

227. Two or more points of a line of position as $(L_1\ \lambda_1)$, (L_2, λ_2) etc., having been determined by Prob. 55, if the true latitude, L, be subsequently found, the corresponding longitude, λ, may be obtained by interpolation.

Or, the place of the ship may be found graphically upon the chart, by drawing a parallel in the latitude, L, and taking its intersection P, with the line of position A B.

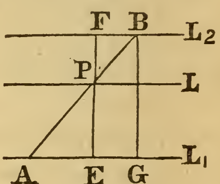

Fig. 39.

So also, if the true longitude, λ, is subsequently found, the corresponding latitude, L, may be obtained by interpolation; or, a meridian E F may be drawn in the longitude, λ, which will intersect the line of position in P, the place of the ship.

If there is uncertainty in either of these elements, two parallels of latitude (as in Fig. 38), or two meridians, may be drawn at a distance apart equal to the uncertainty.

As altitudes, latitudes, and longitudes are never found at sea with much precision, and may under unfavorable circumstances be largely in error, the position of the ship on the chart is not properly a point, but a belt, more or less limited according to the accuracy of the elements from which it has been formed.

228. In Fig. 39, if A is the position (L_1, λ_1),
B, the position (L_2, λ_2),
both near P, the true position, whose latitude is L, and longitude is λ;

the right triangles* A C B, A E P, being formed,

$C B = L_2 - L_1$, the difference of the two latitudes,

$A C = \lambda_2 - \lambda_1$, the difference of the corresponding longitudes,

$E P = \Delta L = L - L_1$, the correction of L_1,

$A E = \Delta \lambda = \lambda - \lambda_1$, the correction of λ_1; then

$$C B : E P = A C : A E$$

or, $(L_2 - L_1) : \Delta L = (\lambda_2 - \lambda_1) : \Delta \lambda$,

whence we have,

$$\left.\begin{aligned} \Delta \lambda &= \Delta L \frac{\lambda_2 - \lambda_1}{L_2 - L_1} \\ \text{and} \qquad \lambda &= \lambda_1 + \Delta \lambda \end{aligned}\right\} \qquad (162)$$

as the formulas for finding λ, the longitude of the true position, when its latitude, L, is known.

Or, we have

$$\left.\begin{aligned} \Delta L &= \Delta \lambda \frac{L_2 - L_1}{\lambda_2 - \lambda_1} \\ \text{and} \qquad L &= L_1 + \Delta L \end{aligned}\right\} \qquad (163)$$

as the formulas for finding L, when λ is given. They are the same formulas as for an interpolation. The several differences are most conveniently expressed in minutes of arc, or, in the case of longitudes, in seconds of time. The

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* This is different from the projection on a Mercator's chart, where G B and E P would be augmented differences of latitude.

local times may be used instead of the longitudes and interpolated in the same way.

From the first of (162) we may readily determine how much a supposed error in an assumed latitude affects the resulting local time, or longitude.

229. PROBLEM 56. *To find from a line of position the azimuth of the body observed.*

Solution. We have the positions (L_1, λ_1), (L_2, λ_2), or the latitudes and longitudes of two points, from which the azimuth, or course of the line of position, can be found by *middle latitude sailing.*

Adding or subtracting 90°, according as the azimuth of the body is greater or less, gives the azimuth required.

Or, a perpendicular to the line of position may be drawn upon the chart, and the angle which it makes with a meridian may be measured with a protractor. The azimuth may thus be found to the nearest 1°.

EXAMPLE.

At sea, 1865, Nov. 23, $10\frac{1}{2}$ A. M., by account in lat. 36° 50′ N., long. 65° 20′ W.; Greenwich mean time $2^h\ 40^m\ 47^s$; the sun's correct central altitude 29° 6′ 25″; to find the line of position.

G. m. t. Nov. 23	$2^h\ 40^m\ 47^s = 2^h.68$	☉'s *dec.*	*Eq'n of t.*
Eq. of t.	+13 18	−20° 25′ 33″ −30″.9	$-13^m\ 20^s.4 + 0^s.72$
G. ap. t.	2 54 5	61 .8	+1 .9 ⎰ 1 .4
		−1 23 ⎨ 18 .7	−13 18 5 ⎱ .5
		−20 26 56 ⎩ 2 .4	

1. *With assumed Latitudes.* (Prob. 43.)

$h =$	29° 6′ 25″	$L_1 = 36°\ 30′$,	$L_2 = 36°\ 50′$,	$L_3 = 37°\ 10′$,
$L_1 =$	36 30 0	l. sec 0.09482	.09670	.09861
$p =$	110 26 56	l. cosec 0.02827	.02827	.02827
$2s =$	176 3 21			
$S =$	88 1 40	l. cos 8.53674	.49841	.45636
$s-h =$	58 55 15	l. sin 9.93271	.93346	.93422
G. ap. t.	$2^h\ 54^m\ 5^s$	18.59254	.55684	.51746

$$\text{L. ap. t.}\begin{cases}(1)\ 22\ 28\ 37 \\ (2)\ 22\ 32\ 27 \\ (3)\ 22\ 36\ 22\end{cases} \qquad \text{l. sin } \underline{9.29677} \qquad \underline{.27842} \qquad \underline{.25873}$$

$$\begin{cases}\lambda_1 = 4\ 25\ 28 = 66^\circ\ 22'.0\ \text{W.} & L_1 = 36^\circ\ 30'\ \text{N.} \\ \qquad\qquad\qquad\qquad -57'.5 & \\ \lambda_2 = 4\ 21\ 38 = 65\ \ 24.5 & L_2 = 36\ \ 50 \\ \qquad\qquad\qquad\qquad -58'.8 & \\ \lambda_3 = 4\ 17\ 43 = 64\ \ 25.7 & L_3 = 37\ \ 10\end{cases}$$

For $\varDelta L = +\,40'$, $\varDelta\lambda = -\,116'.3$; or a change of $40'$ in latitude produces a change of $-116'$ in longitude.

From $\varDelta L = +\,40'$, $\varDelta\lambda = -\,116'.3$, we find, by middle latitude sailing, the dep. $93'.0$, and then the bearing of the line of position, regarded as a rhumb line, which it nearly is, N. $66^\circ.7$ E.; the sun's azimuth therefore is N. $156^\circ.7$ E.

Suppose the correct latitude to be $36^\circ\ 57'$ N., to find the corresponding longitude on the line of position, we have

$$L = \underline{36^\circ\ 57'}\ \text{N.} \qquad L_2 = 36^\circ\ 50'\ \text{N.} \qquad \lambda_2 = 65^\circ\ 24'.5\ \text{W.}$$

$$\varDelta L = \ +7' \qquad \varDelta\lambda = -\,2'.9 \times 7 = -\ \ 20'.6$$

$$\frac{\lambda_3 - \lambda_2}{L_3 - L_2} = -\,\frac{58'.8}{20} = \ -2'.94 \qquad \lambda = \underline{65^\circ\ \ 4'}\ \ \text{W.}$$

2. *With assumed Longitudes.* (Prob. 46.)

G. ap. time	$2^h\ 54^m\ 5^s$		
	$\lambda_1 = \ \ 4\ \ 19\ \ \ 0$	$\lambda_2 = 4^h\ 21^m\ 0^s$	$\lambda_3 = 4^h\ 23^m\ 0^s$
L. ap. time	$t = -\ 1\ \ 24\ \ 55$	$-\ 1\ \ 26\ \ 55$	$-\ 1\ \ 28\ \ 55$
	l. sec t 0.03052	.03201	.03354
$d = -\ 20^\circ\ 26'\ 56''$	l. tan d 9.57155 n	.57155	.57155 n
	l. tan ϕ'' 9.60207 n	.60356 n	.60509 n
	$\phi'' = -\ 21^\circ\ 48'.1$	$-21^\circ\ 52'.2$	$-21^\circ\ 56'.4$
	l. sin ϕ 9.56984 n	.57114 n	.57244 n
	l. cosec d 0.45671 n	.45671 n	.45671 n
$h = 29^\circ\ 6'\ 25''$	l. sin h 9.68703	.68703	.68703
	l. cos ϕ' 9.71358	.71488	.71618
	$\phi' = +\ 58^\circ\ 51'.7$	$+58^\circ\ 45'.5$	$+58^\circ\ 39'.2$

$$\begin{cases} L_1 = 37^\circ\ \ 3'.6\ \text{N.} & L_2 = 36^\circ\ 53'.3\ \text{N.} & L_3 = 36^\circ\ 42'.8\ \text{N.} \\ \lambda_1 = 64\ \ 45\ \ \ \ \text{W.} & \lambda_2 = 65\ \ 15\ \ \ \ \text{W.} & \lambda_3 = 65\ \ 45\ \ \ \ \text{W.}\end{cases}$$

For $\Delta\lambda = +60'$, $\Delta L = -20'.8$. From these, the bearing of the line of position is N. 66°.5 E.

If the correct longitude is 65° 4′ W., to find the corresponding longitude on the line of position, we have

$$\begin{array}{lll} \lambda = \underline{65^\circ\ 4'}\ \text{W.} & \lambda_1 = 64^\circ\ 45'\ \text{W.} & L_1 = 37^\circ\ 3'.6\ \text{N.} \\ & \Delta\lambda = \ \ +19' & \Delta L = -0'.34 \times 19 = -\ 6'.5 \\ \dfrac{L_2 - L_1}{\lambda_2 - \lambda_1} = -\dfrac{10'.3}{30} = & -\ 0'.34 & L = \underline{36^\circ\ 57'}\ \text{N.} \end{array}$$

Two assumed latitudes, or longitudes, would have sufficed, as the altitude is so small.

230. PROBLEM 57. *To find the position of the observer from two altitudes of the same or different bodies, the Greenwich time being known.*

Solution. Find the line of position from each. If the lines are plotted on the chart, their intersection gives the position required; as the lines A B, C D, in Figs. 40 and 41, which intersect in P.

This intersection may also be readily found by computation, when the lines are regarded as straight.

Let L_1 and L_2 be the same assumed latitudes in both computations; and in Figs. 40 and 41,

A, the point (L_1, λ'_1), called the *first* position,

B, the point (L_2, λ'_2),

both derived from the *first* observation;

C, the point (L_1, λ''_1),

D, the point (L_2, λ''_2),

both derived from the *second* observation:

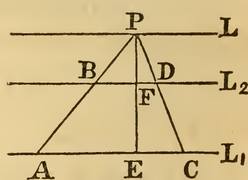

Fig. 40.

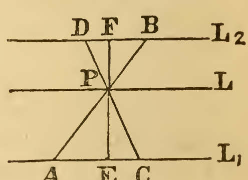

Fig. 41.

the upper accents distinguishing the observations, the lower accents distinguishing the latitude used for each point.

P is the point of intersection, whose latitude L and longitude λ are required.

The diagrams are supposed to be constructed with the differences of latitude and longitude, so that in each

A C $= \lambda''_1 - \lambda'_1$, the diff. of long. in the lat. L_1,
B D $= \lambda''_2 - \lambda'_2$, the diff. of long. in the lat. L_2,
A E $= \Delta\lambda = \lambda - \lambda'_1$, the correction of λ'_1, which it is convenient to call the *first* longitude,
E F $= L_2 - L_1$, the difference of the assumed latitudes,
E P $= \Delta L = L - L_1$, the correction of the *first* latitude.

In Fig. 40, B D lies in the same direction as A C,
F P in the same direction as E F and E P; and
E F = E P − F P.

In Fig. 41, B D is in the opposite direction to A C, and
F P in the opposite direction to E F and E P;
and are therefore to be regarded as negative.

We shall then have, algebraically, in both figures,

$$\text{E F} = \text{E P} - \text{F P}.$$

In the similar triangles A C P, B D P,

$$\text{A C} : \text{B D} = \text{E P} : \text{F P},$$

and, by division,

$$\text{A C} - \text{B D} : \text{A C} = \text{E P} - \text{F P} : \text{E P} = \text{E F} : \text{E P},$$

whence
$$\text{E P} = \frac{\text{A C} \times \text{E F}}{\text{A C} - \text{B D}},$$

or

$$\Delta L = \frac{(\lambda''_1 - \lambda'_1)\,(L_2 - L_1)}{(\lambda''_1 - \lambda'_1) - (\lambda''_2 - \lambda'_2)}.$$

By (162)
$$\Delta\lambda = \Delta L \frac{\lambda'_2 - \lambda'_1}{L_2 - L_1};$$

or substituting for ΔL

$$\Delta\lambda = \frac{(\lambda''_1 - \lambda'_1)\,(\lambda'_2 - \lambda'_1)}{(\lambda''_1 - \lambda'_1) - (\lambda''_2 - \lambda'_2)}.$$

Putting
$$m = \frac{\lambda''_1 - \lambda'_1}{(\lambda''_1 - \lambda'_1) - (\lambda''_2 - \lambda'_2)},$$
we have
$$\left.\begin{aligned} \varDelta L &= m\,(L_2 - L_1), \\ \varDelta \lambda &= m\,(\lambda'_2 - \lambda'_1), \\ L &= L_1 + \varDelta L, \\ \lambda &= \lambda'_1 + \varDelta \lambda, \end{aligned}\right\} \qquad (164)$$

by which the latitude, L, and the longitude, λ, of the intersection can be found.

231. Either assumed latitude may be designated as L_1, and either observation by the accent $'$, or be called the *first* latitude and the *first* observation; but the several differences of latitude and longitude must be marked with their appropriate names, or signs.

If the differences of longitude $\lambda''_1 - \lambda'_1$, $\lambda''_2 - \lambda'_2$, on the two parallels have the same name, their difference is taken in finding m, which will be $+$, when $\lambda''_1 - \lambda'_1 > \lambda''_2 - \lambda'_2$, or the difference of longitude on the first parallel is the greater. In this case $m > 1$, $\varDelta L > L_2 - L_1$ and $\varDelta \lambda > \lambda'_2 - \lambda'_1$. The point P is then, as in Fig. 40, in the same direction as B from A, and beyond B. But m will be $-$, when $\lambda''_1 - \lambda''_1$ and $\lambda''_2 - \lambda'_2$ have the same name, and $\lambda''_1 - \lambda''_1 < \lambda''_2 - \lambda'_2$, or the difference of longitude on the first parallel, is the less. $\varDelta L$ and $\varDelta \lambda$ will then have different names respectively from $L_2 - L_1$ and $\lambda'_2 - \lambda'_1$. In this case P and B are in opposite directions from A. A negative value of m may be avoided, so that P and B will fall always on the same side of A, or P and D always on the same side of C, (Fig. 40), *if we take as L_1 the latitude of the parallel on which is the greatest difference of longitude.*

If the differences of longitude, $\lambda''_1 - \lambda'_1$, $\lambda''_2 - \lambda'_2$, on the two parallels have different names, their sum is taken numerically in finding m; in that case m is $+$ and less than 1, $\varDelta L < L_2 - L_1$ and $\varDelta \lambda < \lambda'_2 - \lambda'_1$, with the same names

respectively; and, as in Fig. 41, P is between A and B, and between C and D.

When three or more latitudes are used in the computations, those for which the differences of longitude are smallest should be taken as L_1 and L_2.

232. The more nearly perpendicular the lines of position are to each other, the better is the determination of their intersection. Hence, the nearer the difference of azimuths of the body or bodies at the two observations is to 90°, the better is the determination of position from double altitudes.

If the azimuths are the same, or differ 180°, the two lines of position coïncide in direction, and there is no intersection. In this case the great circle joining the two bodies, or the two positions of the same body, is an azimuth circle, and passes through the zenith. An approach to this condition is generally to be avoided. (Bowd., pp. 181, 195, notes.) Still, however, if the two bodies, or positions of the same body, are near the meridian, the lines of position nearly coïncide with a parallel of latitude. The latitude is then well determined, but not the longitude. If the two bodies, or positions of the same body, are near the prime vertical, the lines of position more nearly coincide with a meridian and the longitude is well determined, but not the latitude.

When the difference of azimuths is small, the intersection of the two lines may be computed with tolerable accuracy, while it cannot be definitely found by the projection of the lines upon a chart.

233. The operations indicated in (164) are to *subtract*,

1. The *first* assumed latitude from the *second*, (L_2-L_1);
2. for the *first* observation, the longitude corresponding to the *first* latitude from that corresponding to the *second* latitude, $(\lambda'_2-\lambda'_1)$;

3. for *each* latitude, the longitude deduced from the *first* observation from the longitude deduced from the *second*, $(\lambda''_1 - \lambda'_1)$ and $(\lambda''_2 - \lambda'_2)$;
4. the difference of longitude for the *second* latitude from that from the *first*, $[(\lambda''_1 - \lambda'_1) - (\lambda''_2 - \lambda'_2)]$,

(or *add* numerically these differences of longitude when they are of different names.)

Then

5. To divide by this last result the difference of longitude, $(\lambda''_1 - \lambda'_1)$, for the *first* latitude, to obtain the coëfficient m, (which will be — only when the difference of longitude, $(\lambda''_2 - \lambda'_2)$, for the second latitude has the same name as and is greater than the difference of longitude, $(\lambda''_1 - \lambda'_1)$, for the first latitude),
6. To multiply m by the difference, $(L_2 - L_1)$, of the two assumed latitudes to obtain the correction of the first latitude L_1; and by the difference, $\lambda'_2 - \lambda'_1)$, of the two longitudes derived from the first observation, to obtain the correction of the first of these longitudes, λ'_1.

These corrections have the same name as the differences from which they are derived, when m is $+$; but contrary names when m is $-$; and are to be applied accordingly.

234. The lines of position may be found from two assumed longitudes λ_1 and λ_2, instead of two latitudes (Art. 224). The formulas for finding their intersection will differ from (164) only by an interchange of the letters L and λ. We shall have, then,

$$\left.\begin{aligned} m' &= \frac{L''_1 - L'_1}{(L''_1 - L'_1) - (L''_2 - L'_2)} \\ \varDelta\,\lambda &= m' \;(\lambda_2 - \lambda_1), \quad \lambda = \lambda_1 + \varDelta\,\lambda \\ \varDelta\,L &= m' \;(L'_2 - L'_1), \; L = L'_1 + \varDelta\,L \end{aligned}\right\} \quad (165)$$

EXAMPLES.

1. With	$L_1 = 30° \ 28'$ N.,		$L_2 = 30° \ 8'$ N.		Diff. $20'$ S.
by 1st alt.	$\lambda'_1 = 59 \ \ 15$ W.,		$\lambda'_2 = 59 \ \ 0$ W.	"	15 E.
by 2d alt.	$\lambda''_1 = 58 \ \ 43$ W.,		$\lambda''_2 = 59 \ \ 8$ W.	"	25 W.
Differences,	32 E.	—	8 W.	=	40 E.

$$m = \frac{32}{40} = .8$$

$$\varDelta L = -20' \times .8 = -16', \quad L = 30° \ 28' - 16' = 30° \ 12' \text{ N.}$$
$$\varDelta \lambda = -15' \times .8 = -12', \quad \lambda = 59 \ \ 15 - 12' = 59 \quad 3 \text{ W.}$$

The differences of longitude on the two parallels, $32'$ E. and $8'$ W., being in opposite directions, the intersection is between the two parallels, or L is between L_1 and L_2.

2. With	$L_1 = 48° \ 10'$ S.,		$L_2 = 48° \ 30'$ S.		Diff. $20'$ S.
by 1st alt.	$\lambda'_1 = 88 \ \ 16$ E.,		$\lambda'_2 = 88 \ \ 24$ E.	"	8 E.
by 2d alt.	$\lambda''_1 = 88 \ \ 30$ E.,		$\lambda''_2 = 88 \ \ 55$ E.	"	25 E.
Differences,	14 E.,	—	31 E.	=	17 W.

$$m = -\frac{14}{17} = -.82$$

$$\varDelta L = -.82 \times 20' = -16', \quad L = 48° \ 10' - 16' = 47° \ 54' \text{ S.}$$
$$\varDelta \lambda = -.82 \times 8' = -7', \quad \lambda = 88 \ \ 16 - 7' = 88 \quad 9 \text{ E.}$$

In this example it is convenient to regard south latitudes and east longitudes as $+$. The differences of longitude on the two parallels, $14'$ E. and $31'$ E., being in the same direction, the intersection is outside of the parallels and nearer the first, for which we have the smallest difference.

3. With	$\lambda_1 = 165° \ 50'$ W.,		$\lambda_2 = 166° \ 20'$ W.		Diff. $30'$ W.
by 1st alt.	$L'_1 = 36 \ \ 16$ S.,		$L'_2 = 36 \ \ 25$ S.	"	9 S.
by 2d alt.	$L''_1 = 36 \ \ 38$ S.,		$L''_2 = 36 \ \ 29$ S.	"	9 N.
Differences,	22 S.,	—	4 S.	=	18 S.

$$m' = \frac{22}{18} = 1.17$$

$$\varDelta L = 1.17 \times 9' = +10'.5, \quad L = 36° \ 16' + 10' = 36° \ 26' \text{ S.}$$
$$\varDelta \lambda = 1.17 \times 30' = +35'.1, \quad \lambda = 165 \ \ 50 + 35' = 166 \ \ 26 \text{ W.}$$

The differences of latitude on the two meridians, $22'$ S. and $4'$ S., are in the same direction; and the intersection is

outside of the meridians and nearer the second, on which the difference of latitude is least.

235. Problem 57 supposes the two altitudes observed at the same place. This at sea is rarely the case.

PROBLEM 58. *To reduce an observed altitude for a change of position of the observer.*

Solution. Let

Z (Fig. 42) be the zenith of the place of observation;

$h = 90° - Zm$, the observed altitude;

Z′, the zenith of the new position;

$h' = 90° - Z'\,m$, the altitude reduced to the new position, Z'.

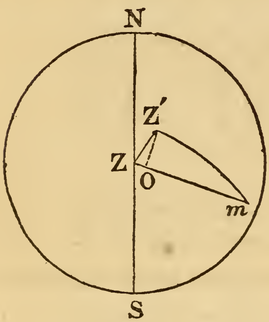

Fig. 42.

$d = Z\,Z'$, the distance of the two places, here referred to the celestial sphere;

$C = P\,Z\,Z'$, the course;

$Z = P\,Z\,m$, the azimuth of m;

$Z - C = m\,Z\,Z'$, the difference of the course and azimuth.

Z Z′, being small, may be regarded as a right line,

Z Z′ O as a plane right triangle, and O m, without material error, as equal to Z′ m; so that we shall have

$$Z\,O = Z\,Z' \cos Z\,Z'\,m$$

$$Z'\,m = Z\,m - Z\,O$$

or, putting $\Delta h = Z\,O$,

$$\left.\begin{aligned} \Delta h &= d \cos (C - Z) \\ h' &= h + \Delta h \end{aligned}\right\} \qquad (166)$$

$\Delta h = d \cos (C - Z)$ is, then, the reduction of the observed altitude to the new position of the observer: it is *additive* when $C - Z < 90°$ numerically; *subtractive* when $C - Z > 90°$. (Bowd., p. 183.) It is smaller, and can,

therefore, be more accurately computed the nearer $C-Z$ approaches 90°. It is, therefore, better to reduce that altitude for which the difference of the course and azimuth is nearest 90°.

If the second is the one reduced, then C is the opposite of the course.

In practice Z Z′ does not usually exceed 30′, so that although an arc of a great circle of the celestial sphere, it may be regarded as representing the distance, d, of the two places on the earth; or, at sea, the distance run. The azimuth, or bearing, of the body can be observed with a compass, or be computed to the nearest degree, or half-degree, from the altitude.

The assumption, $Z' m = O m$, is more nearly correct, the greater $Z' m$ or $Z m$, that is, the smaller the altitude. If we treat $Z Z' m$ as a spherical triangle, $d = Z Z'$ being expressed in minutes and still very small, we shall find

$$\Delta h = d \cos (C-Z) - \tfrac{1}{2} d^2 \sin 1' \tan h \sin^2 (C-Z); \quad (167)$$

but the last term is inconsiderable unless d and h are both large. For instance, if $d = 30'$, it will not exceed 1′ unless $h > 82°$.

Example.

The two altitudes of the sun are 36° 16′ 20″, 58° 15′ 20″, the compass bearings of the sun respectively S. E. by E. ½ E. and W. S. W.; the ship's compass course, and distance made good in the interval N. N. W. ½ W. 25 miles;

S. 5½ E. differs from N. 2½ W. 13 points, so that the reduction of the 1st altitude to the position of the 2d is

$$25' \times \cos 13 \text{ pts.} = -25' \cos 3 \text{ pts.} = -20'.8 = -20'\ 48''.$$

S. 6 W. differs from S. 2½ E. 8½ points, and the reduction of the 2d altitude to the position of the 1st is

$$25' \cos 8\tfrac{1}{2} \text{ pts.} = -25' \cos 7\tfrac{1}{2} \text{ pts.} = -2'\ 30'';$$

or $-2'\ 39''$, if the last term of (167) is included.

236. By (166) or (167) we may reduce one of the two altitudes for the change of the ship's position in the interval. But instead of this we may put down the line of position for each observation, and afterwards move one of them to a parallel position determined by the course and distance sailed in the interval. Thus in Fig. 43, let

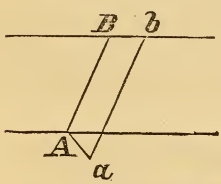

Fig. 43.

A B be the line of position for the first observation, and

A a represent in direction and length the course and distance sailed in the interval; then

a b, drawn parallel to A B, is the line of position which would have been found had the first altitude been observed at the place of the second.

If the second observation is to be reduced to the place of the first, then A a in direction must be the opposite of the course.

The perpendicular distance of A B and a b is the reduction of the altitude for the change of position: for that distance is A $a \times \cos$ (B A $a - 90°$).

LATITUDE BY TWO ALTITUDES.

237. In Sumner's method the latitude and longitude are both found by two altitudes, either of the same or different bodies. It is sometimes desirable to find the latitude only, or at least to make this the chief object of combining the two observations.

238. Problem 59. *To find the latitude from two altitudes of the sun, or other body, supposing the declination to be the*

same at both observations, and the Greenwich time to be known approximately.

Solution. Let two altitudes, or sets of altitudes, be observed and the times noted by a chronometer, or a watch compared with it; reduce the altitudes to true altitudes, and at sea one of them for the change of the ship's position in the interval by Prob. 58. Find also the difference of the chronometer times of the two observations, and correct it for the rate in the interval.

This correction is $\frac{t.\,\varDelta c}{24}$ (+ when the chronometer loses),

t being the interval in hours of chro. time, and
$\varDelta c$ the daily change.

The result is the *elapsed mean time* for a mean time chronometer; the *elapsed sidereal time* for a sidereal chronometer.

The Greenwich mean time of the greater altitude, or of the middle instant, should also be obtained from the chronometer times, sufficiently near for finding the declination of the body.

In Fig. 44, let

M and M' be the two positions of the body,
$h = 90° - Z\,M$, the *first* altitude,
$h' = 90° - Z\,M'$, the *second* altitude,
$d = 90° - P\,M = 90° - P\,M'$, the common declination, and
$t = M\,P\,M'$, the difference of the hour-angles,
$Z\,P\,M$ and $Z\,P\,M'$, of the body at the two observations:

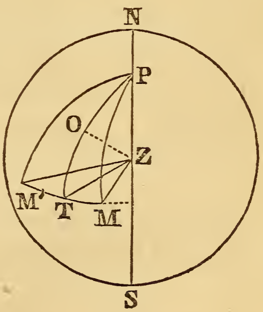

Fig. 44.

or, letting T and T' designate these hour-angles in the order of time,

$$t = T' - T.$$

The method for finding t is different for different bodies.

a. For a *fixed star* the angle M P M′, or t, is the *elapsed sidereal time.* An elapsed mean time must therefore be reduced to the equivalent sidereal interval. If, then,

S and S' represent the sidereal times of the two observations,
t_m and s, the elapsed mean and sidereal times,

we have, when the sidereal interval has been found,

$$t = S' - S = s; \tag{168}$$

when the mean time interval has been found, by (85),

$$t = t_m + .00274\ t_m; \tag{169}$$

or, with t_m expressed in hours in the last term (87),

$$t = t_m + 9^s.8565\ t_m. \tag{170}$$

b. For the *moon* or *a planet*, if

a and $a + \varDelta a$ represent the right ascensions of the body at the two times, we have (Art. 111),

$$T = S - a, \qquad T' = S' - a - \varDelta a,$$

and

$$t = T' - T = s - \varDelta a, \tag{171}$$

that is, the elapsed sidereal time *diminished* by the *increase* of the right ascension of the body in the interval.

$\varDelta_h a$, the change of right ascension in 1^h of mean time, may be obtained from the Almanac for the middle Greenwich time.

The change in 1^h of sidereal time will be, by (86),

$$(1 - .00273)\ \varDelta_h a,$$

which can readily be found by regarding $\varDelta_h a$ as a sidereal interval, and reducing it to its equivalent mean time interval.

Expressing t_m and s in hours, when used as coëfficients, we have

$$\varDelta a = t_m.\ \varDelta_h a = s.\ \varDelta_h a\ (1 - .00273); \tag{172}$$

and, for an elapsed *sidereal* time,

$$t = s - s.\ \varDelta_h a\ (1 - .00273); \tag{173}$$

for an elapsed *mean* time, by (170),

$$t = t_m + t_m\ (9^s.8565 - \Delta_h a). \qquad (174)$$

c. For the *sun*, the angle M P M′, or t, is the elapsed *apparent* time. If, then, $\Delta_h E$ is the hourly change of the equation of time (+ when the equation of time is additive to *mean* time and increasing, or subtractive from *mean* time and decreasing; that is, when apparent time is gaining on mean time),

$$t = t_m + t_m.\ \Delta_h E, \qquad (175)$$

by which t may be found from a mean time interval. If the sidereal interval is given, we have, as in (173), for a planet,

$$t = s - s.\ \Delta_h a\ (1 - .00273).$$

The reduction of the elapsed mean time to an apparent time interval, is commonly neglected by navigators; but on December 21, $\Delta_h E = 1^s.25$, and during a large part of the year exceeds $0^s.5$.

239. We have given

$$h = 90° - \text{Z M}, \qquad d = 90° - \text{P M} = 90° - \text{P M}',$$
$$h' = 90° - \text{Z M}', \qquad t = \text{M P M}',$$

to find from the several triangles of Fig. 44, the latitude,

$$L = 90° - \text{P Z}.$$

Various solutions of this problem have been given, from which the following are selected.

240. (*A.*) 1*st Method in Bowditch's Navigator.* Let T (Fig. 44), be the middle point of M M′; join P T and Z T; and put

A = M T = M′ T = ½ M M′, or half the distance of the two positions of the body,

B = 90° − P T, the declination of T,

H = 90° − Z T, the altitude of T,

q = P T Z, the position angle of T.

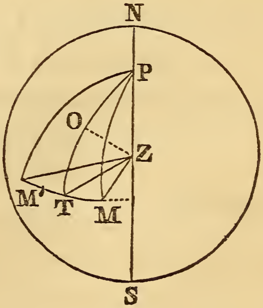

Fig. 44.

As P T M and P T M′ are equal right triangles, we have the angle P T M = P T M′ = 90°,

$$q = 90° - \text{Z T M} = \text{Z T M}' - 90°,$$

and

$$\tfrac{1}{2}\, t = \text{M P T} = \text{M}'\text{P T}.$$

1. In the right triangle P T M, by Sph. Trig. (80), (84), and (82),

$$\left.\begin{aligned} \sin A &= \cos d \sin \tfrac{1}{2}\, t, \\ \sin B &= \sin\, d \sec A; \end{aligned}\right\} \qquad (176)$$

or

$$\tan B = \tan d \sec \tfrac{1}{2}\, t; \qquad (177)$$

by which A and B, or M T and 90° − P T, can be found.

2. In the two triangles Z M T, Z M′ T, by Sph. Trig. (4),

$$\sin h = \sin H \cos A + \cos H \sin A \sin q;$$
$$\sin h' = \sin H \cos A - \cos H \sin A \sin q;$$

the half difference and half sum of which, by Pl. Trig. (106) and (105), are

$$\sin \tfrac{1}{2}\,(h - h') \cos \tfrac{1}{2}\,(h + h') = \cos H \sin A \sin q,$$
$$\cos \tfrac{1}{2}\,(h - h') \sin \tfrac{1}{2}\,(h + h') = \sin H \cos A;$$

from which,

$$\sin H = \frac{\cos \frac{1}{2}\,(h - h') \sin \frac{1}{2}\,(h + h')}{\cos A}, \qquad (178)$$

$$\sin q = \frac{\sin \frac{1}{2}\,(h - h') \cos \frac{1}{2}\,(h + h')}{\cos H \sin A}, \qquad (179)$$

which determine H and q, or 90° − Z T and the angle P T Z.

3. In the triangle P T Z, by Sph. Trig. (4),

$$\sin L = \sin B \sin H + \cos B \cos H \cos q,$$

To adapt this to computation by logarithms, put

$$\left.\begin{aligned} \cos C \sin Z &= \cos H \cos q, \\ \cos C \cos Z &= \sin H, \end{aligned}\right\} \qquad (180)$$

and then

$$\left.\begin{aligned} \tan Z &= \cot H \cos q, \\ \cos C &= \sin H \sec Z, \\ \sin L &= \cos C \sin\, (B + Z), \end{aligned}\right\} \qquad (181)$$

which determine C, Z, and L. If, however, we add the squares of (180), we shall have

$$\cos^2 C = \cos^2 H \cos^2 q + \sin^2 H,$$

or
$$1 - \sin^2 C = \cos^2 H (1 - \sin^2 q) + \sin^2 H;$$

whence

$$\sin C = \cos H \sin q. \tag{182}$$

Substituting this in (179), and the 2d of (180) in (178), we have

$$\left.\begin{aligned} \sin C &= \frac{\sin \frac{1}{2} (h - h') \cos \frac{1}{2} (h + h')}{\sin A}, \\ \cos Z &= \frac{\cos \frac{1}{2} (h - h') \sin \frac{1}{2} (h + h')}{\cos A \cos C}, \\ \text{and (181)} \quad \sin L &= \cos C \sin (B + Z). \end{aligned}\right\} \tag{183}$$

Z, being found from its cosine, may have two values numerically equal with contrary signs. Representing these, we have

$$\sin L = \cos C \sin (B \pm Z),$$

which gives two values of L. The value which accords most nearly with the latitude by account may be taken. We shall see presently how the admissible value of Z may be selected.

241. To avoid using both the sine and cosecant of A and the cosine and secant of C, we may take the reciprocals of (176) and the 2d of (183); we shall then have, as in the 1st method of Bowditch (p. 180),

$$\left.\begin{aligned} \operatorname{cosec} A &= \sec d \operatorname{cosec}^* \tfrac{1}{2} t, \\ \operatorname{cosec} B &= \operatorname{cosec} d \cos A, \\ \sin C &= \sin \tfrac{1}{2} (h - h') \cos \tfrac{1}{2} (h + h') \operatorname{cosec} A, \\ \sec Z &= \sec \tfrac{1}{2} (h - h') \operatorname{cosec} \tfrac{1}{2} (h + h') \cos A \cos C, \\ \sin L &= \cos C \sin (B \pm Z). \end{aligned}\right\} \tag{184}$$

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* log sec $\frac{1}{2} t$ and log cosec $\frac{1}{2} t$ may be taken from Table XXVII. corresponding to t in the column P M (Art. 126).

It is unnecessary to find A and C, as log cosine A can be taken from the tables corresponding to log cosecant A, and log cosine C corresponding to log sine C. Indeed, we may dispense with A entirely by substituting the 1st of the preceding equations in the 3d, and the 2d in the 4th, and employing (177). We shall then have, for the solution of the problem,—

$$\left.\begin{array}{l}\tan B=\tan d \sec^* \tfrac{1}{2}\, t,\\ \sin\ C=\sin \tfrac{1}{2}\,(h-h') \cos \tfrac{1}{2}\,(h+h') \sec d \operatorname{cosec}^* \tfrac{1}{2}\, t,\\ \sec\ Z=\sec \tfrac{1}{2}\,(h-h') \operatorname{cosec} \tfrac{1}{2}\,(h+h') \sin d \operatorname{cosec} B \cos C,\\ \sin\ L=\cos\ C \sin\ (B\pm Z).\end{array}\right\}(185)$$

A, B, C, Z, and L, are each numerically less than 90°;
A is in the 1st quadrant;
C is + when the 1st altitude is the greater, — when it is the smaller;
B has the same sign, or name, as the declination; and
L the same as $(B+Z)$ or $(B-Z)$, from which it is obtained.

242. If Z O (Figs. 44 and 45), be drawn perpendicular to P T, we shall find from (182),

$C=\pm$Z O, + when P T is west, — when it is east, of the meridian; $Z=$T O,
$B+Z=90°-$P O in Fig. 44,
$B-Z=90°-$P O in Fig. 45;

Z, or T O, being + or − according as P and Z are on the same side of M M′, as in Fig. 44, or on opposite sides, as in Fig. 45.

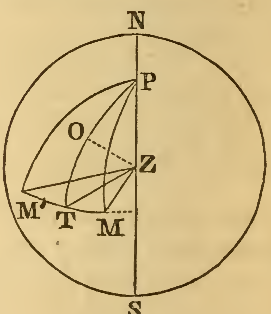

Fig. 44.

This may also be shown in another way: for, in the first case, the

* log sec $\tfrac{1}{2} t$ and log cosec $\tfrac{1}{2} t$ may be taken from Table XXVII. corresponding to t in the column P M (Art. 126).

angle $q = \mathrm{P\,T\,Z} < 90°$, cos q is $+$; and since, in (180), cos C and sin H are $+$, sin Z, and therefore Z, will also be $+$. In the second case, when M M′ produced passes between P and Z, $q = \mathrm{P\,Z\,Z} > 90°$, cos q and, consequently, sin Z and Z are $-$.

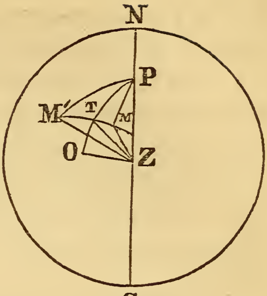

Fig. 45.

Instead of marking Z + or −, we may use the symbols N and S, as for d, B, and L. We shall then have the rule (Bowd., p. 181):—

Mark Z north, or south, according as the zenith and *north* pole, or the zenith and *south* pole, are on the same side of the great circle, which joins the two positions of the body. By thus noting the position of this circle, the ambiguity of Z is removed.

We may, however, remove the ambiguity by noting the azimuths of the two points M and M′.

In Fig. 44, $\mathrm{P\,Z\,M} > \mathrm{P\,Z\,M'}$; in Fig. 45, $\mathrm{P\,Z\,M} < \mathrm{P\,Z\,M'}$; which would be the case also if one or both points were on the other side of the meridian. Hence we have the rule:—

Z has the same name as the latitude when the azimuth of the body is numerically the greater at the greater altitude; but a different name from the latitude when the azimuth at the greater altitude is the less. The azimuths are to be reckoned both east and west from 0 to 180°, and from the N. point in north latitude; but from the S. point in south latitude.

243. If Z is very small, it cannot be accurately found from its cosine, or secant; its sign may be doubtful; and the latitude cannot be determined with precision. This will be the case, when the altitudes are very great; when M and M′ are near the prime vertical; or, in general, when M and M′ are remote from the meridian and the difference of azimuths,

M Z M′, is either very small, or near 180°. In each of these cases M M′ intersects the meridian very near the zenith.

It has been seen, with regard to lines of position derived from two altitudes (Art. 232), that the most favorable condition is when M Z M′ = 90°; but that the latitude alone can be well determined when M and M′ are quite near the meridian in azimuth and M Z M′ quite small. Indeed, if both azimuths are 0, or 180°, the two altitudes become a meridian altitude.

These conditions belong to all methods of finding the latitude from two altitudes.

244. The latitude having been found, we may proceed to find the hour-angle of the body from one of the altitudes (Prob. 43), if it is sufficiently near the prime vertical, and thence the longitude, if the times have been carefully noted by a Greenwich chronometer (Art. 188).

Instead of this, by putting $T_0 = \frac{1}{2}(T + T') = $ Z P T, the middle hour-angle, we have the formula (146),

$$\sin T_0 = \frac{\sin \frac{1}{2}(h - h') \cos \frac{1}{2}(h + h')}{\cos d \cos L \sin \frac{1}{2} t}; \qquad (186)$$

and from (185)

$$\sin C = \frac{\sin \frac{1}{2}(h - h') \cos \frac{1}{2}(h + h')}{\cos d \sin \frac{1}{2} t};$$

whence

$$\sin^* T_0 = \frac{\sin C}{\cos L}. \qquad (187)$$

This could also have been obtained from the right triangle P O Z, (Figs. 44, 45); from which we have also

$$\tan \mathrm{Z\,P\,O} = \frac{\tan \mathrm{Z\,0}}{\sin \mathrm{P\,O}},$$

or

$$\tan T_0 = \frac{\tan C}{\cos (B \pm Z)} \qquad (188)$$

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* If we enter Table XXVII. (Bowd.) with log sin T_0, or log tan T_0, we shall find 2 T_0 corresponding in the P. M. column.

Thus, by a brief additional computation, T_0 can be found by (186) or (187). We shall have also

$$T = T_0 - \tfrac{1}{2}\, t, \qquad T' = T_0 + \tfrac{1}{2}\, t \qquad (189)$$

for the hour-angles at the times of the observations. The longitude can be found from either T_0, T, or T', and the corresponding chronometer time.

(186) is the formula of Littrow's method* (Art. 200). The favorable conditions, as there stated, for finding T_0, are a small value of T_0 and high altitudes near the meridian, or altitudes on each side of the meridian near the prime vertical. But such altitudes are unfavorable for finding the latitude.

When both latitude and longitude are to be found from two altitudes, the nearer the difference of azimuths is to 90° the better will be the determination. The most favorable conditions for combining them will be equal azimuths of 45°, or 135°, on each side of the meridian.

If one of the altitudes is very near the prime vertical, and the other very near the meridian, it will generally be better to find the time and longitude from the first by Prob. 43, and the latitude from the second by Prob. 46 or 47.

245. In this problem the declinations are supposed to be the same at both observations. This will be the case with the sun only at the solstices; with a planet, or the moon, only when 90° from its node, and with the latter body for a very brief period. Navigators usually neglect the change of declination of the sun, or a planet, and use the mean declination, or that for the middle instant. It is better, however, when the change is neglected, to employ the declination at the time of the greater altitude,† except when the

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* The novelty of Littrow's method consists in finding T_0 from very high altitudes near the meridian. (146), or (186), is by no means a new formula.

† Chauvenet's Astronomy, Vol. I., pp. 276, 315.

hour-angle of this altitude is greater than the middle hour-angle. This can be the case only when the altitudes are on different sides of the meridian. When the middle declination is used, we may, with little additional labor, find the correction of the computed latitude by the following formula from Chauvenet's Astronomy (Vol. I., p. 267).

$$\Delta L = -\frac{\Delta d. \sin C}{\cos L \sin \frac{1}{2} t} \tag{190}$$

or, by substituting (187)

$$\Delta L = -\frac{\Delta d. \sin T_0}{\sin \frac{1}{2} t} \tag{191}$$

in which Δd is *half* the change of declination in the interval of the observations. Noting whether it is toward the north or the south, we can apply it with the *same* name to the computed latitude, when the *lesser* altitude was observed *first;* but with a *different* name when the *lesser* altitude was observed *last.*

With this correction the preceding method can be employed for altitudes of the moon at sea, when the elapsed time does not exceed 1^h.

The correction of T_0, the middle hour-angle, may also be found by the formula (Chauvenet's Astronomy, Vol. I., p. 268,)

$$\Delta T_0 = \Delta d. \left(\frac{\tan L \cos T_0}{15 \sin \frac{1}{2} t} - \frac{\tan d}{15 \tan \frac{1}{2} t}\right); \tag{192}$$

which differs from the equation of equal altitudes (136) only in the first term being multiplied by $\cos T_0$, and in changing the signs, as it is here applied to the hour-angle instead of the chronometer time.

(B.) *Douwes's Method; Bowditch's 2d Method: with an assumed latitude.*

246. This method differs from the preceding in first finding T_0, the middle hour-angle, by using an approximate lati-

tude, and then the latitude from the greater altitude and its computed hour-angle, as in Problem 46 for finding the latitude from a single altitude.

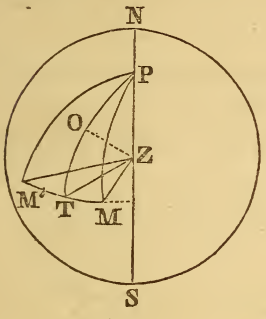

Fig. 46.

Letting L' = the assumed latitude, and as before

$T_0 = \frac{1}{2}(T' + T) = \text{Z P T}$ (Fig. 46) the middle hour-angle.

$t = \frac{1}{2}(T' - T) = \text{M P M}'$, the difference of the hour-angles,

we may use the formulas of Art. 244,

$$\left.\begin{aligned} \sin C &= \frac{\sin \frac{1}{2}(h - h') \cos \frac{1}{2}(h + h')}{\cos d \sin \frac{1}{2} t} \\ \sin T_0 &= \frac{\sin C}{\cos L} \\ T = T_0 - \tfrac{1}{2} t&, \; T' = T_0 + \tfrac{1}{2} t \end{aligned}\right\} \qquad (193)$$

and of Art. 149,

$$\left.\begin{aligned} \cos z_0 &= \sin h + 2 \cos d \cos L' \sin^2 \tfrac{1}{2} T \\ \cos z_0 &= \sin h' - 2 \cos d \cos L' \sin^2 \tfrac{1}{2} T' \end{aligned}\right\} \qquad (194)$$

selecting that which contains the greater altitude and less hour-angle.

$z_0 = 90° - h_0$ is the meridian zenith distance from which the latitude may be found as from an observed meridian altitude. (Prob. 45.)

Should this differ from the assumed latitude, the computations should be repeated, using this new value.

The method can be used to most advantage when T_0 is small, and the greater altitude is observed near the meridian. But it has the inconvenience of requiring several recomputations, when the computed latitude differs widely from that assumed. If the observations are unfavorable for

finding the latitude, successive recomputations will approximate very slowly, or may not approximate at all, to a conclusive result. When the less altitude is near the prime vertical, it is preferable to find from it the greater hour-angle, and then the less by adding or subtracting t.

247. Mr. Douwes's formulas, however, are somewhat different.

From the triangles Z P M, Z P M′ we have by Sph. Trig. (4)

$$\sin h \;= \sin d \sin L + \cos d \cos L \cos T,$$
$$\sin h' = \sin d \sin L + \cos d \cos L \cos T'\,;$$

the difference of which is

$$\sin h - \sin h' = \cos d \,\cos L\,(\cos T - \cos T').$$

This by Pl. Trig. (130) becomes

$$\sin h - \sin h' = 2 \cos d \,\cos L \,\sin \tfrac{1}{2}\,(T + T) \sin \tfrac{1}{2}\,(T' - T')\,;$$

whence, since

$$T_0 = \tfrac{1}{2}\,(T + T') \text{ and } \tfrac{1}{2}\,t = \tfrac{1}{2}\,(T' - T),$$
$$2 \sin T_0 = (\sin h' - \sin h) \sec d \sec L \operatorname{cosec} \tfrac{1}{2}\,t, \qquad (195)$$

which gives T_0, provided for L we use the approximate latitude L'.

We then have as before

$$T = T_0 - \tfrac{1}{2}\,t,\; T' = T_0 + \tfrac{1}{2}\,t \qquad (196)$$

and for finding the meridian zenith distance from the greater altitude, (116)

$$\left.\begin{array}{l} \cos z_0 = \sin h + \cos L' \cos d \operatorname{versin} T, \\ \text{or,} \\ \cos z_0 = \sin h' + \cos L' \cos d \operatorname{versin} T', \end{array}\right\} \qquad (197)$$

Mr. Douwes prepared special tables (Tab. XXIII., Bowd.,) to facilitate the use of formulas (195) and (197) calling

log cosec $\tfrac{1}{2}\,t$,	log of "half elapsed time,"
log (2 sin T_0),	log of "middle time,"
log versin t,	"log rising";

increasing the indices of the last two by 5, so that natural sines, etc., to 5 decimal places may be treated as whole numbers.

(193) and (194) are preferable, as they require only tables, which are in common use.

248. When the declination is taken for the middle time, we have for the correction of the computed latitude (191)

$$\varDelta L = -\frac{\varDelta d. \sin T_0}{\sin \frac{1}{2} t} \qquad (198)$$

in which $\varDelta d$ is the change of declination in *half* the interval of time between the observations.

EXAMPLE.

At sea, 1865, April 16, A. M. and P. M., in lat. 37° 20′ S., long. 12° 3′ W., by account, two altitudes of the sun were observed for finding the latitude, the corrected times and altitudes being as follows:

G. m. t. April	15 $23^h\ 16^m\ 25^s$	☉'s true alt. 38° 16′ 25″,	S. $15\frac{1}{2}$ pts. E.
" "	16 1 18 26	" 42 15 30,	S. $16\frac{1}{4}$ W.
Mid. G. m. t. "	16 0 17 25.5	$\frac{1}{2}(h+h')=$ 40 15 57	
—Eq. of t.	+16.4	$\frac{1}{2}(h-h')=-1$ 59 32	
Mid. G. ap. t.	0 17 42		
Elapsed *m. t.*	2 2 1	☉'s dec., $d=+10°\ 13'\ 57''$	
Ch. of eq. of t.	+1	Ch. in 1^h	+53.04
Elapsed ap. t.	$t=$ 2 2 2	Ch. in $1^h.017$	$\varDelta d=+53.7$

Computation by (185), (187) *and* (191).

l. sec $\frac{1}{2}t$	0.01558	l. cosec $\frac{1}{2}t$	0.58001		
l. tan d	9.25651	l. sec d	0.00696	l. sin d	9.24955
l. tan B	9.27209	l. cos $\frac{1}{2}(h+h')$	9.88256	l. cosec $\frac{1}{2}(h+h')$	0.18954
		l. sin $\frac{1}{2}(h+h')$	8.54113 n	l. sec $\frac{1}{2}(h-h')$	0.00026
$B=+10°\ 35'\ 52''$		l. sin C	9.01066 n	l. cos C	9.99771
$Z=-47$ 45 25		l. cos C	9.99771	l. cosec B	0.73539
$B+Z=-37$ 9 33		l. sin $(B+Z)$	9.78106 n	l. sec Z	0.17245

$L_1 = -36\ 55\ 52$	l. sin L	$9.77877\,n$	l. sec L	0.09726
$\Delta L = \quad +26$	mid. G.ap.t. =	$0^h\ 17^m\ 42^s$	l. sin C	$9.01066\,n$
$L = 36\ 55\ 26$ S.	T_0 =	$-0\ 29\ 28$	l. sin T_0	$9.10792\,n$
	Long.	$+0\ 47\ 10$	l. cosec $\frac{1}{2}\,t$	0.580
	or	$11°\ 47'\ 30''$ W.	log $\Delta\, d$	1.730
			log $(-\Delta\, L)$	$1.418\,n$

The azimuth at the greater altitude being the greater, Z is —, or S., like the latitude.

Computation by (193), (194) *and* (191), *with assumed lat.* 37° 20′ *S.*

$t = 2^h\ 2^m\ 2^s$	l. cosec $\frac{1}{2}$	0.58001		
$\frac{1}{2}(h+h') = 40°\ 15'\ 57$	l. cos	9.88256		
$\frac{1}{2}(h-h') = -\ 1\ 59\ 32$	l. sin	$8.54113\,n$	log 2	0.30103
$d = +10\ 13\ 57$	l. sec	0.00696	l. cos d	9.99304
C	l. tan	$9.01066\,n$	l. 2 cos d	0.29407
$L' = -37\ 20$	l. sec	0.09957	l. cos L'	9.90043
$T_0 = -0^h\ 29^m\ 37^s$	l. sin	$9.11023\,n$	2 l. sin $\frac{1}{2}\,T'$	7.67078
$\frac{1}{2}\,t = +1\ 1\ 1$				
$T' = +0\ 31\ 24$		733	log	7.86528
$h' = 42°\ 15'\ 30''$	sin	$.67248$		
$z_0 = -47\ 10\ 25$	cos	$.67981$		
$d = 10\ 13\ 57$				
$L_1 = -36\ 56\ 28$				

2d Approximation.

C	l. tan	$9.01066\,n$	l. 2 cos d	0.29407
$L_1 = -36°\ 56'\ 28''$	l. sec	0.09732	l. cos L''	9.90268
$T_0 = -\ 0^h\ 29^m\ 28^s$	l. sin	$9.10798\,n$		
$\frac{1}{2}\,t = +\ 1\ 1\ 1$	sin h'	$.67248$	2 l. sin $\frac{1}{2}\,T'$	7.67488
$T' = +\ 0\ 31\ 33$		744	log	7.87163
$z_0 = -47\ 9\ 46$	cos	$.67992$	l. sin T_0	$9.108\,n$
$d = +10\ 13\ 57$			l. cosec $\frac{1}{2}\,t$	0.580
$L_2 = -36\ 55\ 49$	Mid. G.ap.t. =	$-0^h\ 17^m\ 42^s$	log $\Delta\, d$	1.730
$\Delta L = \quad +\ 26$	T_0 =	$-0\ 29\ 28$	log $(-\Delta\, L)$	$1.418\,n$
$L = 36\ 55\ 23$ S.	Long.	$+0\ 47\ 10$		
	or	$11°\ 47'\ 30''$ W.		

(C.) *Method of Equal Altitudes.*

249. When the altitudes are equal, or $h=h'$, the equations of (185) become,

$$\left.\begin{aligned} C &= 0 \\ \tan B &= \tan d \sec \tfrac{1}{2} t \\ \cos Z &= \frac{\sin h \sin B}{\sin d} \\ L &= B \pm Z \end{aligned}\right\} \qquad (199)$$

which are identical with (113) for finding the latitude from a single altitude.

We have also $T=-\frac{1}{2}t$, $T'=\frac{1}{2}t$, $T_0=0$; and from (194)

$$\cos z_0 = \sin h + 2 \cos L' \cos d \sin^2 \tfrac{1}{4} t, \qquad (200)$$

from which the meridian zenith distance may be found by one or more approximations.

L having been found by using the declination for the time of meridian passage needs no correction for a small change of declination, since in (198) $\sin T_0 = 0$.

(D.) *Chauvenet's Method, by two altitudes near the meridian when the time is not known.* (Astronomy, Vol. I., p. 296.)

250. The method of reducing circum-meridian altitudes to the meridian, when the time is known, has already been given, (Prob. 47). At sea, however, the local time is frequently uncertain; while altitudes near the meridian are resorted to as next in importance to meridian altitudes for finding the latitude.

As in Prob. 47, let h_0 represent the meridian altitude,

$$\varDelta_0 h = \frac{1''.96349 \cos L \cos d}{\sin (L-d)},$$ the change of altitude in 1^m from the meridian, (Tab. XXXII., Bowd.,)

and as before,

h and h', the true altitudes,

T and T', the corresponding hour-angles, (in minutes of time,)

$t = T' - T$, the difference of the hour-angles,

$T_0 = \frac{1}{2}(T' + T)$, the middle hour-angle.

By (120)

$$\left.\begin{aligned} h_0 &= h + \varDelta_0 h.\ T^2, \\ h_0 &= h' + \varDelta_0 h.\ T'^2. \end{aligned}\right\} \qquad (201)$$

The mean of these equations is

$$h_0 = \tfrac{1}{2}(h + h') + (T'^2 - T^2).\ \varDelta_0 h. \qquad (202)$$

But

$$T'_2 - T^2 = \left(\frac{T' - T}{2}\right)^2 + \left(\frac{T' + T}{2}\right)^2 = (\tfrac{1}{2}t)^2 + T_0^2$$

which substituted in (202) gives

$$h_0 = \tfrac{1}{2}(h + h') + [\tfrac{1}{2}t^2 + T_0^2]\ \varDelta_0 h. \qquad (203)$$

The difference of the two equations of (201) gives

$$h - h' = (T'^2 - T^2).\ \varDelta_0 h = 2\ T_0 t.\ \varDelta_0 h.$$

Hence

$$T_0 = \frac{\frac{1}{2}(h - h')}{t.\ \varDelta_0 h} = \frac{\frac{1}{4}(h - h')}{\frac{1}{2}t.\ \varDelta_0 h}. \qquad (204)$$

Substituting this in (203) we have

$$h_0 = \tfrac{1}{2}(h + h') + (\tfrac{1}{2}t)^2.\ \varDelta_0 h + \frac{[\frac{1}{4}(h - h')]^2}{(\frac{1}{2}t)^2.\ \varDelta_0 h}. \qquad (205)$$

The reduction to the meridian, then, is effected "by adding to the mean of the two altitudes two corrections; 1st, the quantity $(\frac{1}{2}t)^2.\ \varDelta_0 h$, which is nothing more than the *common reduction to the meridian* (120) computed with the half-elapsed time as the hour-angle; 2d, the square of one fourth the difference of the altitudes divided by the first correction." Several pairs of altitudes can be thus combined, and the mean of the meridian altitudes taken, from which the latitude can be obtained as from an observed meridian altitude.

251. The restriction of the method corresponds with that of circum-meridian altitudes (Art. 150).* Quite accurate results can be obtained with hour-angles limited to 5^m when the altitude is 80°, to 25^m when the altitude is only 10°. If the interval t, however, exceed 10^m, $\Delta_0 h$ should be computed to two or three places of decimals, as it is given in Table XXXII. (Bowd.) only to the nearest $0''.1$.

The accuracy of the method depends mainly upon the accuracy of the 2d correction, and therefore upon the precision with which the difference of altitudes has been obtained. The altitudes, then, should be observed with great care. Errors of the tabulated dip and refraction, and a constant error of the instrument will affect both altitudes nearly alike.

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* Note to Art. 150 (omitted in its proper place).

From (117) we have with more exactness,

$$\Delta h = \frac{\cos L \cos d}{\sin (L-d)} \cdot \frac{2 \sin^2 \frac{1}{2} t}{\sin 1''},$$

or putting

$$m = \frac{2 \sin^2 \frac{1}{2} t}{\sin 1''} \quad \text{and} \quad A = \frac{\cos L \cos d}{\sin (L-d)},$$

$$\Delta h = A\, m \quad \text{and} \quad h_0 = h + A\, m.$$

Delambre's formula, obtained by developing in series the preceding equation, (116) is

$$h_0 = h + A\, m - B\, n,$$

in which

$$n = \frac{2 \sin^4 \frac{1}{2} t}{\sin 1''}, \qquad B = A^2 \tan (L-d)$$

Table V., of Chauvenet's Astronomy, contains m and n, and Table VI. contains $\log m$ and $\log n$ for different values of t from 0 to 30^m.

Table VII. A gives the limiting hour-angle at which the error resulting from neglecting the 2d reduction, $B\, n$, amounts to $1''$. It varies from 0 in the zenith to 36^m in latitude 40°, or to 67^m in latitudes 0° and 80°, for an altitude of 10°.

Table XXXII. (Bowd.) gives $\Delta_0 h$ only to the nearest $0''.1$; if, then, it is taken from this table, $\Delta_0 h.\ t^2$ may be in error $1''$, if $t > 4^m$. If, however, $\Delta_0 h$ is computed to the nearest $0''.001$, the error of using $\Delta_0 h.\ t^2$, instead of $A\, m$, will not exceed $1''$, unless $t > 20^m$ and $h > 60°$.

If the altitudes are equal, this second correction becomes 0. The most favorable condition is, therefore, that of equal altitudes observed on each side of the meridian.

At sea, the method is especially useful for altitudes of the sun observed with a clear, distinct horizon. A long interval between the observations is to be avoided on account of the uncertainty of the reduction of one of the altitudes for the run of the ship.

252. The hour-angle of either altitude may also be obtained approximately; for we have from (204), in minutes,

$$\left.\begin{aligned} &T_0 = \frac{\frac{1}{4}(h-h')}{\frac{1}{2}t.\ \Delta_0 h}, \\ \text{and} \qquad &T = T_0 - \tfrac{1}{2}t, \qquad T' = T_0 + \tfrac{1}{2}t. \end{aligned}\right\} \tag{206}$$

Example.

1865, March 14, near noon, in lat. 45° 30′ S., long. 120° 10′ E., by account, two altitudes were observed for latitude,

T. by Chro. $4^h\ 15^m\ 20^s$; Obs'd alt. $\underline{\odot}$ 46° 45′ 30″, (North)
" " " 4 26 16 " " " 46 54 40;

Index cor. of sextant +5′ 20″; height of eye 18 feet.

The sun's declination at noon −2° 31′ 57″, H. ch. +59″.

By preliminary computation $\Delta_0 h = 2''.02$, $\log \Delta_0 h = 0.306$.

$t =$	$10^m\ 56^s$	$h-h' =$	$-9'\ 10''$
$\frac{1}{2}t =$	$5\ \ 28 = 5^m.47$	$\frac{1}{4}(h-h') =$	$-\ 137$
$(\frac{1}{2}t)^2 =$	29.9	$\frac{1}{2}(h+h') =$	46° 50′ 5″
$\Delta_0 h =$	$2''.02$	1st cor. $29.9 \times 2''.02 =$	+1 0
$2 \log \frac{1}{4}(h-h')$	4.274	2d cor. $\frac{137^2}{60.4}$ $=$	+5 11
log (1st cor.)	1.781		46 56 16
log (2d cor.)	2.493	In. cor.	+5 20
$\log \frac{1}{4}(h-h')$	2.137 n	S. diam.	+16 7
ar. co. $\log \frac{1}{2}t$	9.262	Dip	−4 11
ar. co. $\log \Delta_0 h$	9.694	Ref. and par.	−48
$\log T_0$	1.093	$h_0 =$	47 12 44
$T_0 =$	$-\ 12^m.4$	$z_0 =$	42 47 16 S.
$T =$	$-\ 17\ .9$	(12^m before 0^h) $d =$	2 32 9 S.
$T' =$	$-\ 6\ .9$	$L =$	45 19 25 S.

(*E*) *Prestel's method,* by the rate of change of altitude near the prime vertical.*

253. In the note to Art. 197 we have, for a very brief interval of time, and a small change of altitude,

$$\varDelta t = \frac{\varDelta h}{15 \cos L \sin Z},$$

or, using the notation of this problem,

$$T' - T = t = \frac{h' - h}{15 \cos L \sin Z};$$

whence

$$\cos L = \frac{h' - h}{15\, t} \operatorname{cosec} Z; \qquad (207)$$

in which $h' - h$ is expressed in seconds of arc, and t in seconds of time, and, Z being + when *east*, − when *west*, $\cos L$ is always positive. If Z is near 90°, its cosecant varies slowly. When $Z = 90°$, we have,

$$\cos L = \frac{h' - h}{15\, t}. \qquad (207')$$

If, then, two altitudes are carefully observed near the prime vertical, and the times noted with great precision, the interval not exceeding 8 or 10 minutes, an approximate latitude may be found by (207′), when the altitudes are within 2° or 3° of the prime vertical; or by (207) when they are at a greater distance, and Z is approximately known.

The time of passing the prime vertical can be found by (107). Z may be roughly computed from the altitudes, or found within 2° from the bearing observed by a compass, which will suffice, if the observations are made within 10° of the prime vertical.

As, near the prime vertical, the altitude changes uniformly with the time, several altitudes may be observed in quick succession and the mean taken as a single altitude.

The larger $h' - h$ and t, consistent with the supposition of uniformity of change and the condition by which they are

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* Chauvenet's Astronomy, I., pp. 303, 311.

substituted for their trigonometric functions, the more accurate in general will be the result.

Still the method does not admit of much precision. It is entirely unavailable near the equator, and in latitude 45° may give a result in error from 5 to 10 minutes, even when the greatest care has been bestowed on the observations. It may, however, be useful to the navigator in high latitudes, as it can be used for altitudes of the sun, when it is almost exactly east or west, and consequently when no other method is practicable. There are occasions at sea, when to find the latitude only within 10′ is very desirable.

Examples.

1. 1865, June 15, 7^h A. M., in lat. 60° N., lon. 60° W.;

T. by Chro.	$11^h\ 13^m\ 25^s.3$,	obs'd alt. $\underline{\odot}$ 27° 0′ 23″	☉'s Az.
" " "	11 19 51 .0,	" " " 27 48 42	N. 88° E.;

required the latitude.

$\frac{1}{15}$	log	8.8239
$h'-h = 48'\ 19''$	log	3.4622
$t = 6^m\ 25^s.7$	ar. co. log	7.4137
$Z = 88°$	l. cosec	0.0003
$L = \underline{59°\ 55'}$ N.	l. cos	$\underline{9.7001}$

If $\Delta\,(h'-h) = 10''$, $\Delta \log\,(h'-h) = \Delta$ l. cos $L = .0015$, and $\Delta\,L = 6'$. If the difference of altitudes can be depended on within 5″, the latitude is correct within 3′.

2. 1865, July 13, 5^h P. M., in lat. 54° 20′ N., long. 113° W., by account; the altitude of the sun's lower limb was observed at $0^h\ 23^m\ 34^s$ by the chronometer, which was slow of G. mean time $10^m\ 18^s$; and the sextant remaining clamped the upper limb arrived at the same altitude at $0^h\ 27^m\ 8^s.5$; the true altitude of both limbs was 27° 18′ 20″; required the latitude.

The sun's diameter, 31′ 33″, is the difference of altitudes in this case. The sun's azimuth computed from the altitude and supposed latitude is N. 88½° W.

$\frac{1}{15}$	log	8.8239
$h - h' = 31'\ 33''$	log	3.2772
$t = 3^m\ 34^s.5$	ar. co. log	7.6686
$Z = 88\tfrac{1}{2}°$	l. cosec	0.0002
$L = 53°\ 56'$ N.	l. cos	9.7699

If we suppose t to be in error 1^s, l. cos L will be in error .0020 and L, 11′. If the elapsed time can be depended on within $0^s.5$, the latitude is correct within 6′.

The longitude obtained from the same observations is 113° 5′ W.

This method of observing the successive contacts of the two limbs of the sun with the horizon with the sextant clamped is recommended.

254. Problem 60. *To find the latitude from two altitudes of different bodies, or of the same body when the change of declination is considerable, the Greenwich times being known.*

Solution. The observed altitudes should be reduced to true altitudes, watch times to chronometer times, and the difference of the two chronometer times for the rate in the interval, as in Problem 59 (Art. 238), and to a *sidereal* interval, when the altitudes of two different bodies have been observed.

When the latitude only is to be found, the Greenwich mean times of the observations are wanted only with sufficient exactness for finding the right ascensions and declinations of the bodies. If the longitude is also to be found, it is necessary to note the times by a Greenwich chronometer, or a watch compared with it.

It is well also to note the azimuth, or bearing, for each observation; or, as is sufficient, the difference of the azimuths.

Let M and M′ be two positions of the body, or bodies,

$h = 90° - Z\,M$, the true altitude of M,
$h' = 90° - Z\,M'$, the true altitude of M′,
$d = 90° - P\,M$, the declination of M,
$d' = 90° - P\,M'$, the declination of M′,
$T = Z\,P\,M$, the hour-angle of M,
$T' = Z\,P\,M'$, " " " M′,
$t = T' - T = M\,P\,M'$, the difference of the hour-angles.

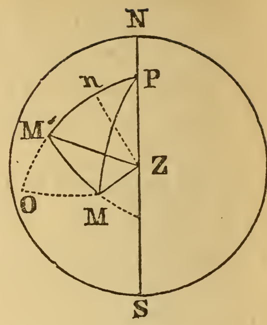

Fig. 47.

t is positive in the direction of the diurnal rotation, and will be positive and less than 12^h if M P M′, estimated from P M in that direction, is less than 12^h, or 180°. We shall, as in the diagram, designate the two positions as M and M′ respectively, so as to satisfy that condition. It will be seen hereafter, however, that it will be sufficient to have t numerically less than 12^h, without regard to its sign.

255. The method of finding t varies with the objects observed. But in any case we are at liberty to add or to subtract 24^h, either to change a negative into a positive result, or to reduce it within the numerical limit of 12^h. A positive result greater than 12^h, or a negative result less than 12^h indicates that P M′ is in the negative direction from P M.

a. When *two bodies* are observed at *different* times, if

a and a' are their right ascensions,
S and S', the sidereal times of the observations,

by Art. 111,

$$T = S - a, \qquad T' = S' - a'$$

and
$$t = S' - S - a' + a;$$

or,
$$t = s + a - a' \tag{208}$$

when M has been first observed;

$$t = -(s + a' - a) \qquad (208')$$

when M′ has been first observed.

In either case, the right ascension of the body first observed is added to the sidereal interval, and the right ascension of the other body subtracted. If M′ has been observed first, the sign is to be changed.

b. When *two bodies* are observed at the *same* time,

$$s = 0 \text{ and } t = a - a', \qquad (209)$$

the difference of their right ascensions.

c. When the *sun*, *moon*, or *a planet* is observed at two different times, we have, as in Art. 238,

$$t = s - s.\ \Delta_h\, a\ (1 - .00273) \qquad (210)$$

for an elapsed *sidereal* time;

$$t = t_m + t_m\ (9^s.8565 - \Delta_h\, a) \qquad (211)$$

for an elapsed *mean* time; in which $\Delta_h\, a$ is the change of right ascension in 1^h of mean time; $(1 - .00273)\ \Delta_h\, a$, the change of right ascension in 1^h of sidereal time; and t_m and s as coëfficients are expressed in hours.

In the case of the sun the last expression becomes, as in (175)

$$t = t_m + t_m.\ \Delta_h\, \mathrm{E};$$

in which $\Delta_h\, E$ is the hourly change of the equation of time, employing for E the sign of its application to *mean* time.

If this result exceed 12^h, it should be subtracted from 24^h. M′ in that case is the position at the first observation.

256. We have given (Fig. 47)

$$h = 90° - \mathrm{Z\,M}, \qquad d = 90° - \mathrm{P\,M},$$
$$h' = 90° - \mathrm{Z\,M'} \qquad d' = 90° - \mathrm{P\,M'},$$
$$\text{and } t = \mathrm{M\,P\,M'},$$

to find from the triangles P M M′, Z M M′ and P M′ Z, or P M Z,

$$L = 90° - \text{P Z}.$$

The following method is selected as the most common.

257. *Fourth Method of Bowditch's Navigator.*

1. In the triangle P M M′, (Fig. 48)

$$\text{M P M}' = t, \qquad \text{P M} = 90° - d, \qquad \text{P M}' = 90° - d',$$

are given, from which we may find

M M′ = B, the distance of the two positions, and the angle, P M′ M = P′,

By Sph. Trig. (4) and (10)* we have

$$\cos B = \sin d' \sin d + \cos d' \cos d \cos t,$$

$$\cot P' = \frac{\cos d' \tan d - \sin d' \cos t}{\sin t};$$

the 2d of which, by multiplying both numerator and denominator by cos d, becomes

$$\cot P' = \frac{\cos d' \sin d - \sin d' \cos d \cos t}{\cos d \sin t}.$$

To adapt these to logarithmic computation, put

$$\left.\begin{array}{l} m \sin M' = \sin d \\ m \cos M' = \cos d \cos t \end{array}\right\}$$

and we shall have, after eliminating m,

$$\left.\begin{array}{l} \tan M' = \tan d \sec t, \\ \cos B = \dfrac{\sin d \cos (M' - d')}{\sin M'}, \\ \cot P' = \dfrac{\cot t \sin (M' - d')}{\cos M'} \end{array}\right\} \quad (212)$$

---

* $\text{Cos } a = \cos b \cos c + \sin b \sin c \cos A,$ (4)

$\sin A \cot B = \sin c \cot b - \cos c \cos A$ (10)

We may take M' in the same quadrant as t and give to it the sign of d. $\frac{\sin d}{\sin M'}$ is then positive, and B will be in the 1st or 2d quadrant as the numerical value of $M'-d'$. $\frac{\cot t}{\cos M'}$ is also positive, and P' will be in the 1st or 2d quadrants according as $M'-d'$ has the positive or negative sign.

If M O be drawn perpendicular to P M', (Fig. 47) we shall find $90° - \text{P O} = M'$, the declination of O, and $\text{M}' \text{O} = M' - d'$.

2. In the triangle Z M M',

$$\text{Z M} = 90° - h, \qquad \text{Z M}' = 90° - h', \qquad \text{M M}' = B,$$

being known, if we put Z M' M $= Q'$, we have, by Sph. Trig.* (30)

$$\sin \tfrac{1}{2} Q' = \sqrt{\left(\frac{\sin \frac{1}{2}(90° - h + 90° - h' + B) \sin \frac{1}{2}(90° - h - 90° + h' - B)}{\cos h' \sin B}\right)}$$

and by reducing

$$\sin \tfrac{1}{2} Q' = \sqrt{\left(\frac{\cos \frac{1}{2}(B + h' + h) \sin \frac{1}{2}(B + h' - h)}{\cos h' \sin B}\right)};$$

or, putting

$$\left.\begin{aligned} s &= \tfrac{1}{2}(B + h' + h), \\ \sin \tfrac{1}{2} Q' &= \sqrt{\left(\frac{\cos s \sin (s - h)}{\cos h \sin B}\right)}; \end{aligned}\right\} \tag{213}$$

Since this radical may have either the positive or negative sign, we may take $\frac{1}{2} Q'$ in either the 1st or 4th quadrants, or numerically less than 90° with either sign. We shall thus have two equal values of Q' with opposite signs.

3. P M' Z = P M' M − Z M' M (Fig. 47), or representing it by q',

$$q' = P' - Q';$$

for which we shall have two values, resulting from the two values of Q', which are indicated by

$$q' = P' \mp Q'. \tag{214}$$

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* $$\sin^2 \tfrac{1}{2} A = \frac{\sin \frac{1}{2}(a + b - c) \sin \frac{1}{2}(a - b + c)}{\sin b \sin c} \tag{30}$$

In the triangle P M′ Z,

$$\text{P M}' = 90° - d', \quad \text{Z M}' = 90° - h', \quad \text{P M}'\text{Z} = q',$$

being known, we have, by Sph. Trig. (4), for finding

$$L = 90° - \text{P Z},$$
$$\sin L = \sin h' \sin d' + \cos h' \cos d' \cos q'.$$

To adapt this for logarithms, put

$$\left.\begin{aligned} n' \sin N' &= \cos h' \cos q' \\ n' \cos N' &= \sin h', \end{aligned}\right\}$$

and we shall have, after eliminating n',

$$\left.\begin{aligned} \tan N' &= \cot h' \cos q' \\ \sin L &= \frac{\sin h' \sin (N' + d')}{\cos N'}. \end{aligned}\right\} \qquad (215)$$

We may take N' numerically less than 90°, and give it the same sign as that of $\cos q'$: and the latitude L, numerically less than 90°, with the same sign as $N' + d'$.

There will be two values of L derived from the two values of q'. Unless q' is small, we may select the value which agrees best with the known approximate latitude.

If Z n (Fig. 47) be drawn perpendicular to P M′, we shall find M′ $n = N'$ and $90° - \text{P}\, n = N' + d'$, the declination of n.

258. Thus, by (212), (213), (214), and (215), the solution is effected. We have seen in each how the proper quadrant of the unknown quantities can be determined (with the restriction of t to positive values less than 12^{h}), except that Q' may have two values. The same results would be obtained by following the usual trigonometric precepts.

259. We may, however, select the proper value of Q' and avoid the double solution, by means of the noted azimuths. For, if Z and Z' are the azimuths of M and M′ (Fig. 47), reckoned as positive toward the right,

$$\text{M Z M}' = Z' - Z,$$

and in the triangle M Z M′

$$\sin Q' = \sin (Z' - Z) \frac{\cos h}{\sin B}.$$

As $\cos h$ and $\sin B$ are positive, Q' will have the same sign as $(Z' - Z)$ restricted numerically to 180°. Hence, as Q' is to be subtracted from P', we shall have

$q' = P' - Q'$, when M′ bears to the *right* of M,
$q' = P' + Q'$, when M′ bears to the *left* of M.

Figs. 47 and 48 illustrate these two cases, for, in the first,

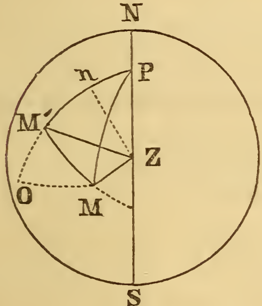

Fig. 47.

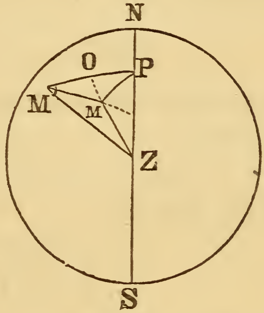

Fig. 48.

where M′ is to the right of M, P M′ Z = P M′ M − Z M′ M; and in the second, where M′ is to the left of M, P M′ Z = P M′ M − Z M′ M.

If M M′ be extended to the meridian, in the first case, P and Z are on the same side of the intersection, and in the other they are on opposite sides; so that

$q' = P' - Q'$ when the zenith and *north* pole are on the same side of the great circle, which joins the two positions;

$q' = P' + Q'$, when the zenith and *north* pole are on different sides, or the zenith and *south* pole are on the same side, of that circle.

To use this criterion it will be necessary to note where the circle, which connects the two positions observed, crosses the meridian.

The doubtful case with either of these criterions is when $Z'-Z$ is near 0, or 180°, or the great circle, which joins the two positions, passes near the zenith. These two conditions are coïncident, except when the two positions are near the meridian on the same side of the zenith.

260. The hour-angle of M′ may also be found, and thence the longitude, if the times have been noted by a chronometer regulated to Greenwich time. For, in the triangle P M′ Z, we have, by Sph. Trig. (10),

$$\cot T' = \frac{\cos d' \tan h' - \sin d' \cos q'}{\sin q'},$$

which, by multiplying numerator and denominater by $\cos h$, becomes

$$\cot T' = \frac{\cos d' \sin h' - \sin d' \cos h' \cos q'}{\cos h' \sin q'}.$$

Putting, as before,

$$n' \sin N' = \cos h' \cos q'$$
$$n' \cos N' = \sin h',$$

and eliminating n', we have

$$\left.\begin{aligned} \tan N' &= \cot h' \cos q' \\ \cot T' &= \frac{\cot q' \cos (N'+d')}{\sin N'}. \end{aligned}\right\} \tag{216}$$

Or, L having been found, we have also

$$\sin T' = \frac{\sin q' \cos h'}{\cos L}. \tag{217}$$

By (217) $\sin T'$ and $\sin q'$ have the same sign, which will be positive when M′ is west of the meridian, negative when M′ is east of the meridian. In (216), if N' has been taken less than 90° with the same sign as that of $\cos q'$, $\frac{\cos q'}{\sin N'}$ is positive, and $\cos T'$ and $\cos (N'+d')$ have the same sign. We may take T', then, numerically in the same quadrant as $N'+d'$, and give it the positive sign when $q' < 180°$, the negative sign when $q' > 180°$, or is negative.

If the proper value of Q', and therefore q', has not been

previously determined, we shall have two values of T', but may ordinarily take that which agrees best with its known approximate value.

261. The preceding formulas employ the angles at M′, and the triangle P M′ Z. We may also use the angles at M and the triangle P M Z; and, as M and M′ are similarly situated with regard to the triangles, except that the angles at each are estimated in opposite directions, we shall obtain, by interchanging accented and unaccented letters in the preceding formulas, a set similar in form, but with this difference of interpretation, that t is positive in the opposite direction of the diurnal rotation, and q is less than 180° east of the meridian and greater than 180°, or negative, west of the meridian. This difference is shown in Fig. 49, in which the primitive position of the triangles is *east* of the meridian, instead of *west* as in Fig. 47.

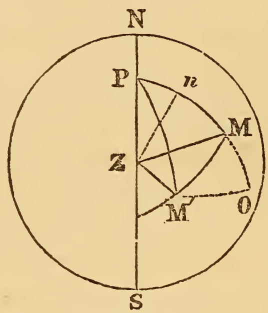

Fig. 49.

We have, then,

$$\left.\begin{aligned} \tan M &= \tan d' \sec t, \\ \cos B &= \frac{\sin d' \cos (M - d)}{\sin M}, \\ \cot P &= \frac{\cot t \sin (M - d)}{\cos M}, \end{aligned}\right\} \qquad (212')$$

$$\left.\begin{aligned} s &= \tfrac{1}{2}(B + h + h'), \\ \sin \tfrac{1}{2} Q &= \sqrt{\left(\frac{\cos s \sin (s - h')}{\cos h' \sin B}\right)}, \end{aligned}\right\} \qquad (213')$$

$$q = P \mp Q, \qquad (214')$$

$$\left.\begin{aligned} \tan N &= \cot h \cos q, \\ \sin L &= \frac{\sin h \sin (N + d)}{\cos N}, \end{aligned}\right\} \qquad (215')$$

$$\cot T = -\frac{\cot q \cos (N + d)}{\sin N}, \qquad (216')$$

$$\sin T = -\frac{\sin q \cos h}{\cos L}. \qquad (217')$$

$q = P - Q$, when M bears to the left of M′,
$q = P + Q$, when M bears to the right of M′,

262. Either set of formulas may be used; but, in general, the latitude can be best found from the altitude, which is nearest the meridian; the hour-angle, from the altitude which is nearest the prime vertical.

The distinction made with regard to M and M′ (Art. 254), is important only so far as it may aid in determining the hour-angles and selecting the proper value of q or q'. So that it is sufficient practically to find t numerically less than 12^h without regard to its sign.

263. The most favorable condition is, as stated in Article (243), when the difference of azimuths is 90°. But altitudes near the meridian will give a good determination of the latitude, and altitudes near the prime vertical, a good determination of the hour-angles, when the difference of azimuths is small, or near 180°; especially if the altitudes have been carefully observed, and their difference is nearly exact.

264. If we put in (212′ &c.)

$$M = -A, \qquad B = C, \qquad P = 90° - F,$$
$$Q = Z, \qquad q = 90° - G, \qquad N = I,$$

we shall have

$$\left.\begin{aligned}
\tan A &= -\tan d' \sec t, \\
\cos C &= -\frac{\sin d' \cos (A+d)}{\sin A}, \\
\cot F &= -\frac{\tan t \cos A}{\sin (A+d)}, \\
s &= \tfrac{1}{2}(C+h+h'), \\
\sin \tfrac{1}{2} Z &= \sqrt{\left(\frac{\cos s \sin (s-h')}{\cos h' \sin B}\right)}, \\
G &= F \pm Z, \\
\tan I &= \cot h \sin G, \\
\sin L &= \frac{\sin h \sin (d+I)}{\cos I};
\end{aligned}\right\} \qquad (218)$$

the formulas of Bowditch's 4th method, if we take h' and d' as the greatest altitude and the corresponding declination.

By attending to the signs of the quantities and their functions, the proper result can be obtained. We may give to Z the *same* sign, or name, as that of the latitude, when the zenith and elevated pole are on the *same* side of the great circle, which passes through the two positions observed; but a *different* sign, or name, from that of the latitude, when the zenith and elevated pole are on *opposite* sides of that circle.

The precepts, which Bowditch gives (p. 194) are based on the consideration of the trigonometric functions, and possess the advantage that the *sum* of quantities of the *same* name, or the *difference* of quantities of *different* names, is taken, and the name of the greater given to the result. If the sum exceed 180°, it should be subtracted from 360°, and the name changed.

Example.

At sea, 1865, May 5, 7^h P. M., in lat. 36° 41′ N., long. 168° 57′ W., by account; altitudes of a Ursæ Majoris and the moon were observed and the means noted as follows: required the latitude and longitude.

T. by Chro. $6^h 41^m 27^s$; alt. of ✱ 62° 18′ 30″; bearing N. by E. ½ E.
" " " 6 53 8; " " ☽ 44 56 50; " S. E. ¾ S.
Chro. at 18^h fast of G. m. t. $36^m 48^s$, losing daily $10^s.2$;
Index cor. of sextant —4′ 20″; height of eye 18 feet.
Ship running N. E. by N. (true), 10 knots an hour.

From these data, reducing the second altitude to the position of the first, we find,

	a Ursæ Maj.	*Moon.*		
G. m. t. May 5	$18^h\ 4^m\ 39^s.3$	$18^h\ 16^m\ 20^s.4$	Elap. m. t.	$11^m\ 41^s.1$
R. A.	10 55 24 .3	11 37 57 .0	Red.	+1 .9
Dec.	$d' = +62°\ 28'\ 45''$	$d = -0°\ 53'\ 54''$	—Elap. sid t.	—11 43 .0
True alt.	$h' =$ 62 9 29	$h =$ 45 39 54	Diff. of R. A.	42 32 .7
				$t =$ 30 49 .7

Computation by (212–216).

$t=$	7° 42′ 26″	l. sec t	0.00394		
$d=-$	0 53 54	l. tan d	8.19535 n	l. sin d	8.19530 n
$M'=-$	0 54 24	l. tan M'	8.19929 n	l. cosec M'	1.80077 n
$d'=+$	62 28 45				
$M'-d'=-$	63 23 9			l. cos $(M'-d')$	9.65126
$B=$	63 38 39	l. cosec B	0.04766	l. cos B	9.64733
$h=$	45 39 54			l. cot t	0.86859
$h'=$	62 9 29	l. sec h'	0.33066	l. sec M'	0.00005
$2s=$	171 28 2			l. sin $(M'-d')$	9.95136 n
$s=$	85 44 1	l. cos s	8.87153	l. cot P'	0.82000 n
$s-h=$	40 4 7	l. sin $(s-h)$	9.80869		
			19.05854		
$\frac{1}{2}Q'=$	19 46 19	l. sin $\frac{1}{2}Q'$	9.52927		
$Q'=$	39 32 38				
$P'=$	171 23 36	l. cot h'	9.72278		
$q'=$	210 56 14	l. cos q'	9.93335 n	l. cot q'	0.22230
or	−149 3 46	l. tan N'	9.65613 n	l. cosec N'	0.38441 n
$N'=-$	24 22 20	l. sin h'	9.94657	l. cos $(N'+d')$	9.89590
$d'=+$	62 28 45	l. sec N'	0.04054	l. cot T'	0.50261 n
$N'+d'=+$	38 6 25	l. sin $(N'+d')$	9.79038	$T'=-1^h\ 9^m\ 48^s$	
		l. sin L	9.77749	✳'s R.A.=	10 55 24.3
				Sid. time	9 45 36.3
Lat.	36° 48′.3 N.			$-S_0$	−2 53 28.5
Long.	168 52.4 W.			−Red. for G. m. t.	−2 58.2
				L. m. t.	6 49 9.6
				G. m. t.	18 4 39.3
				Long.	11 15 29.7

265. When *equal altitudes* have been observed, (213) and (213′) reduce to the simple form,

$$\sin \tfrac{1}{2} Q' = \sin \tfrac{1}{2} Q = \sqrt{\left(\frac{\cos\left(\frac{1}{2}B+h\right)}{2\cos\frac{1}{2}B\cos h}\right)}. \qquad (219)$$

We have also from the isosceles triangle Z M M′ (Fig. 47),

$$\cos Q' = \cos Q = \tan \tfrac{1}{2} B \tan h. \qquad (220)$$

266. When a lunar distance has been measured and reduced to the *true* or *geocentric* distance (Prob. 55), we have in the triangle P M M′ (Fig. 47),

$$\text{P M} = 90° - d, \quad \text{P M}' = 90° - d', \quad \text{and}$$
$$\text{M M}' = B, \text{ the } \textit{geocentric} \text{ distance},$$

from which we may find P M′ M = P′ and P M M′ = P.

By Sph. Trig.* (30),

$$\sin \tfrac{1}{2} P' = \sqrt{\left(\frac{\sin \frac{1}{2}(90° - d + 90° - d' - B) \sin \frac{1}{2}(90° - d - 90° + d' + B)}{\sin B \cos d'}\right)}$$

which reduces to

$$\sin \tfrac{1}{2} P' = \sqrt{\left(\frac{\cos \frac{1}{2}(B + d + d') \sin \frac{1}{2}(B + d' - d)}{\sin B \cos d'}\right)}$$

or putting

$$\left.\begin{aligned} s' &= \tfrac{1}{2}(B + d + d'), \\ \sin \tfrac{1}{2} P' &= \sqrt{\left(\frac{\cos s' \sin (s' - d)}{\sin B \cos d'}\right)} \end{aligned}\right\} \tag{221}$$

So also we shall have

$$\sin \tfrac{1}{2} P = \sqrt{\left(\frac{\cos s' \sin (s' - d')}{\sin B \cos d}\right)}. \tag{221'}$$

These may be employed instead of (212) and (212′); and if the altitudes of both bodies have been observed, the latitude and hour-angles can be found by the subsequent formulas.

From a lunar distance, then, and the two observed altitudes, the longitude, latitude, and local time may all be found.

267. Problem† 61. *To find the latitude from three altitudes of the same body near the meridian, and the chronometer times of the observations.*

The Greenwich time, or the longitude, will be required only with sufficient exactness for taking the declination from the Ephemeris.

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* $$\sin^2 \tfrac{1}{2} A = \frac{\sin \frac{1}{2}(a + b - c) \sin \frac{1}{2}(a - b + c)}{\sin b \sin c}. \tag{30}$$

† Chauvenet's Astronomy, Vol. I., p. 299.

Let h, h', h'', be the true altitudes,
h_0, the meridian altitude,
T, T', T'', the chronometer times,
T_0, the chronometer time of meridian passage,
a, the change of altitude in 1^s of chronometer time from the meridian,

then we have, from the three observations (120), the differences of the times being expressed in seconds,

$$\left.\begin{aligned} h_0 &= h + a\,(T - T_0)^2 \\ h_0 &= h' + a\,(T' - T_0)^2 \\ h_0 &= h'' + a\,(T'' - T_0)^2 \end{aligned}\right\} \qquad (222)$$

the differences of which are

$$0 = h' - h + a\,[(T' - T_0)^2 - (T - T_0)^2],$$
$$0 = h'' - h' + a\,[(T'' - T_0)^2 - (T' - T_0)^2].$$

From these we obtain

$$\left.\begin{aligned} \frac{h - h'}{T' - T} &= a\,(T' + T) - 2\,a\,T_0, \\ \frac{h' - h''}{T'' - T'} &= a\,(T'' + T') - 2\,a\,T_0, \end{aligned}\right\} \qquad (223)$$

the difference of which is

$$\frac{h' - h''}{T'' - T'} - \frac{h - h'}{T' - T} = a\,(T'' - T). \qquad (224)$$

If now we put

$b = \frac{h - h'}{T' - T}$, the mean change of altitude in 1^s of the chronometer from the first to the second observation,

$c = \frac{h' - h''}{T'' - T'}$, the mean change in 1^s from the second to the third observation,

we shall have, from (224) and (223),

$$\left.\begin{aligned} &a = \frac{c - b}{T'' - T}, \\ &T_0 = \tfrac{1}{2}\,(T + T') - \frac{b}{2\,a}, \quad \text{or} \quad T_0 = \tfrac{1}{2}\,(T' + T'') - \frac{c}{2\,a}, \end{aligned}\right\} \qquad (225)$$

from which a and T_0 may be found. h_0 may then be found by either of the three equations (222); and thence the latitude as from an observed meridian altitude (Prob. 45).

The correction and rate of the chronometer need not be known; it is sufficient to have the rate uniform during the period of the observations.

This method is restricted like other methods of circum-meridian altitudes (Arts. 150, 247), to hour-angles varying with the meridian zenith distance of the body. Its accuracy depends upon the precision with which a, b, and c are obtained: hence the altitudes should be carefully observed, so that their differences shall be nearly exact, and the intervals of time should be greater, the greater the distance of the middle observation from the meridian.

268. The computation is facilitated if the observations are made at intervals of exact minutes of time. For then, expressing these intervals in minutes, and taking a, b, and c as changes of altitude in 1^m of chronometer time,

$$b = \frac{h - h'}{T' - T}, \qquad c = \frac{h' - h''}{T'' - T'}, \qquad a^* = \frac{c - b}{T'' - T} \tag{226}$$

are easily found. $\frac{b}{2\,a}$ and $\frac{c}{2\,a}$, however, will be in minutes of time. Reducing them to seconds, we shall have, instead of (225),

$$T_0 = \tfrac{1}{2}(T + T') - \frac{30\,b}{a} \quad \text{or} \quad T_0 = \tfrac{1}{2}(T' + T'') - \frac{30\,c}{a}. \tag{227}$$

In (222)

$$\left.\begin{aligned} h_0 &= h + a\,(T - T_0)^2 \\ h_0 &= h' + a\,(T' - T_0)^2 \\ h_0 &= h'' + a\,(T'' - T_0)^2 \end{aligned}\right\}$$

we may now use, in computing the reductions, a table of

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* If the intervals are reduced by the methods of Art. 238 to intervals of hour-angle, and the declination has not changed, a will be the change of altitude in 1^m of hour-angle, as in (119) and Tab. XXXII. (Bowd.).

"squares of minutes and parts of a minute" as Tab. XXXIII. (Bowd.).

269. For the sun, as its declination usually changes in the interval, T_0 is the chronometer time of the *maximum* altitude* (Art. 142) ; in which case the *meridian* declination is to be employed.

If the latitude and longitude also have changed, as is usually the case at sea, c and b, being observed changes of altitude, are no longer due to the diurnal rotation alone, but are affected by the change of position. But with altitudes near the meridian, a change of latitude has the same effect as a change of declination in the opposite direction, while a change of longitude is equivalent to a change in the rate of the chronometer. If, then, the motion of the ship has been tolerably uniform in the interval of the observations, T_0 is still the chronometer time of the *maximum altitude.* The method, then, can be used at sea when the sea is smooth and the horizon well defined, and meridian altitudes of the sun are prevented by passing clouds. But the altitudes should be very carefully observed, and on both sides of the meridians when practicable. The intervals should not be less than 10^m.

270. If the three altitudes are observed at *equal* intervals of time, the process of computation becomes much simplified.†

Let t be this common interval,

T, the time from the maximum altitude at which the second observation was made ;

then we have

$$h_0 = h + a\,(T - t)^2$$
$$h_0 = h' + a\,T^2$$
$$h_0 = h'' + a\,(T + t)^2.$$

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* Chauvenet's Astronomy, Vol. I., pp. 299 and 244.

† Chauvenet's Astronomy, Vol. I., p. 309.

Half the sum of the second and third equations is

$$h_0 = \tfrac{1}{2}(h + h'') + a\,(T^2 + t^2),$$

which, subtracted from the third, gives

$$0 = h' - \tfrac{1}{2}(h + h'') - a\,t^2;$$

whence

$$a\,t^2 = h' - \tfrac{1}{2}(h + h'').$$

The difference of the first and third gives

$$a\,T = \frac{\tfrac{1}{4}(h - h'')}{t}$$

$$a\,T^2 = \frac{[\tfrac{1}{4}(h - h'')]^2}{a\,t^2},$$

which, substituted in the second equation, gives h_0.

If we put $A = a\,t^2$, we have, as the formulas for computation,

$$\left.\begin{aligned} A &= h' - \tfrac{1}{2}(h + h'') \\ h_0 &= h' + \frac{[\tfrac{1}{4}(h - h'')]^2}{A} \end{aligned}\right\} \qquad (228)$$

Examples.

1. At sea, 1865, Sept. 16, in lat. 40° 0′ N., long. 60° 0′ W., by account; the following altitudes of the sun were observed near noon; index cor. +2′ 10″; height of eye 20 feet.

T. by Chro.	*Diff.*	Obs'd alt. of $\odot$	*Diff.*
$4^h\ 25^m\ 15^s$		52° 24′ 20″ (S.)	
	7^m		4′ 10″
32 15		20 10	
	5		5 20
37 15		14 50	

$\tfrac{1}{2}(T + T') = 4^h\ 28^m\ 45^s$	$b = \dfrac{250''}{7} = 35''.7$	$\log b$	1.553
$-\dfrac{30\,b}{a} = \ -7\ \ 34$	$c = \dfrac{320''}{5} = 64\ .0$	$\log 30$	1.477
$T_0 = 4\ \ 21\ \ 11$	$a = \dfrac{28''.3}{12} = 2\ .36$	ar. co. $\log a$	9.627
$T = 4\ \ 25\ \ 15$	454^s	$\log \dfrac{30\,b}{a}$	2.657
$T - T_0 = \ \ 4\ \ 4$	$a(T - T_0)^2 = 2''.36 \times 16.5 = 39''$		

	1st alt. of ☉	52° 24′ 20″	In. cor.	+2′ 10″	dip	−4′ 24″
		+13 45	S. diam.	+15 58	ref. & par.	−38
Mer. alt.	$h_0 =$	52 38 5	Red.	+39		
Mer. zen. dist.	$z_0 =$	37 21 55 N.				
Mer. dec.	$d =$	2 28 48 N.				
	Lat.	39 50 43 N.				

If the middle altitude had been 30″ greater or less than 52° 20′ 10″, the result would have been varied only 20″; but if the middle altitude had been 1′ less, the latitude would have been 39° 40′. The interval is too short, unless the differences of the altitudes can be relied on within 40″.

2. At sea, 1865, May 8, in lat. 35° 50′ S., long. 60° 0′ E., by account; the following altitudes of the sun were observed at equal intervals near noon: index cor. +2′ 0″; height of eye 20 feet.

Obs'd alt. of ☉	36° 44′ 20″ (N.)		T. by watch	$11^h\ 50^m\ 20^s$	
	36 51 40			12 0 20	
	36 52 40			12 10 20	
$\frac{1}{2}(h+h') =$	36 48 30		$h-h'' =$	−8′ 20″	
$A =$	3 10		$\frac{1}{4}(h-h'') =$	− 125″	2 log 4.194
					log A 2.279
					1.915

	$h' =$	36° 51′ 40″	Red.	+1′ 22″		
		+13 42	In. cor.	+2 0	Dip	−4′ 24″
Mer. alt.	$h_0 =$	37 5 22	S. diam.	+15 53	Ref. & par.	−1 9
Mer. zen. dist.	$z_0 =$	52 54 38 S.				
Mer. dec.	$d =$	17 7 19 N.				
	Lat.	35 47 19 S.				

If either of the altitudes be changed 1′, the reduction to the meridian will be changed less than 40″: so that, if the differences can be depended on within 1′, the reduction is correct within 40″.

CHAPTER X.

AZIMUTH OF A TERRESTRIAL OBJECT.

271. In conducting a trigonometric survey, it is necessary to find the azimuth, or true bearing, of one or more of its lines, or of one station from another. Thence, by means of the measured horizontal angles, the azimuths of other lines or stations can be found; and, still further, a meridian line can be marked out upon the ground, or drawn upon the chart.

For example, suppose at a station, A, the angles reckoned to the *right* are

B to C, 48° 15′ 35″; C to D, 73° 37′ 16″; D to E, 59° 45′ 20″;

and that the azimuth of D is N. 35° 16′ 15″ E.; the azimuths of the several lines are

$A\ B$,	N. 86°	36′	36″	W.	$A\ D$,	N. 35°	16′	15″	E.
$A\ C$,	N. 38	21	1	W.	$A\ E$,	N. 95	1	35	E.

If upon the chart a line be drawn, making with $A\ B$ an angle of 86° 36′ 36″ to the *right*, or with $A\ D$ an angle of 35° 16′ 15″ to the *left*, it will be a meridian line.

Or, if a theodolite or compass be placed at A in the field, and its line of sight, through the telescope or sight-vanes, be directed to D, and the readings noted; and then the line of sight be revolved to the *left* until the readings differ 35° 16′ 15″ from those noted, it will be directed *north*. If a stake or mark be placed in that direction, it will be a *meridian mark* north from A.

272. If the azimuth of a terrestrial object is known, it may be conveniently used in finding the magnetic declination, or variation of the compass. For, let the bearing of the object be observed with the compass,—the difference of this magnetic bearing and the true bearing is the magnetic declination, or variation, required. It is *east* if the true bearing is to the *right* of the magnetic bearing; but *west* if the true bearing is to the *left* of the magnetic bearing.*

273. Problem 62. *To find the azimuth, or true bearing, of a terrestrial object.*

Solution. Let

Z (Fig. 50) be the zenith, or place, of the observer;

O, the terrestrial object;

M, the *apparent* place of the sun, or some other celestial body;

$Z = $ N Z O, the azimuth of O;

$z = $ N Z M, the azimuth of M;

$\zeta = Z - z = $ M Z O, the *azimuth angle* between the two objects, or the difference of azimuth of M and O.

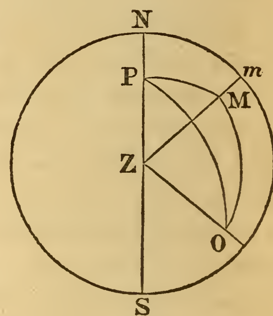

Fig. 50.

The problem requires that z and ζ be found; then we have

$$Z = z + \zeta.$$

Or, numerically,

---

* This has reference to the two readings. The actual direction of the object is the same; but the true and magnetic meridians, from which the angles are estimated, are different. When the magnetic declination is *east*, the magnetic meridian is to the *right* of the true meridian; when the magnetic declination is *west*, the magnetic meridian is to the *left* of the true meridian.

It is sometimes necessary to distinguish between the magnetic bearing and the compass bearing. The latter is affected by the errors of the instrument employed and by local disturbances; the former is free from them.

$Z = z + \zeta$, when the azimuth of the terrestrial object is greater than that of the celestial,

$Z = z - \zeta$, when it is less. The sign of ζ should be noted in the observations.

274. $z =$ N Z M, the azimuth of the celestial body, may be found from an observed altitude (Prob. 40), or from the local time (Prob. 38). In the first case, the most favorable position is on or nearest the prime vertical; for then the azimuth changes most slowly with the altitude. In the latter, positions near the meridian may also be successfully used.

275. $\zeta =$ M Z O, the azimuth angle between the two objects, may be found in one of the following ways:—

1*st Method.* (By direct measurement.)

M Z O, being a *horizontal* angle, may be measured directly by a theodolite or a compass, by directing the line of sight of the instrument first to one of the objects and reading the horizontal circle, then to the other and reading again. The difference of the two readings is the angle required. Or, the telescope or sight-vanes of a plane table may be directed successively to the objects, and lines drawn upon the paper along the edge of the ruler in its two positions, and the angle which they form measured by a protracter.

At the instant when the observation is made of the celestial object, either its altitude should be measured, or the time noted, so as to find its azimuth simultaneously.

The instrument should be carefully adjusted and levelled. With the compass or plane table, it is not well to observe objects whose altitudes are greater than 15°.

A theodolite can be used with greater precision than the other instruments; but the greater the altitude of the object, the more carefully must the cross-threads be adjusted to the axis of collimation, and the telescope be directed to the object.

The *error of collimation* is eliminated by making two observations with the telescope reversed either in is Vs, or by rotation on its axis. Low altitudes are generally best.

276. If the sun is used, each limb may be observed alternately; or a separate set of observations may be made for each.

To find the azimuth reduction for semi-diameter, when but one limb is observed;

Let $h = 90° - $ Z s (Fig. 51), the altitude of the sun,

$s =$ S s, its semi-diameter,

$s' =$ S Z s, the reduction of the azimuth for the semi-diameter.

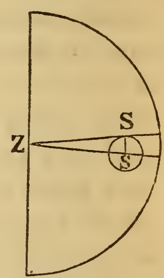

Fig. 51.

We have

$$\sin \text{S Z s} = \frac{\sin \text{S s}}{\sin \text{Z s}},$$

or, since s and s' are small,

$$s' = s \sec h, \tag{229}$$

which is the reduction required.

The sign, with which it is to be applied, depends upon the limb observed.

277. If the observations are made at night, and the terrestrial object is invisible, a temporary station in a convenient position may be used, and its azimuth found. The horizontal angle between this and the terrestrial object may be measured by daylight, and added to, or subtracted from this azimuth.

A board, with a vertical slit and a light behind it, forms a convenient mark for night observations.

The place of the theodolite should be marked, that the instrument may be replaced in the same position. But in doing this, and selecting the temporary station, it should be kept in mind that a change of the position of the instru-

ment of $\frac{1}{3438}$ of the distance of the object may change the azimuth 1′; or of $\frac{1}{200000}$ of the distance may change the azimuth more than 1″.

278. 2d *Method.* Finding the difference of azimuths of a celestial and a terrestrial object *by a sextant;* sometimes called an "*astronomical bearing.*"

Measure with a sextant the angular distance M O (Fig. 52) of the two objects, and either note the time by a watch regulated to local time, or measure simultaneously the altitude of the celestial object. Measure, also, the altitude of the terrestrial object (if it is not in the horizon), either with a theodolite which is furnished with a vertical circle, or with a sextant above the water-line at the base of the object, when there is one. Correct the readings of the instruments for index errors, and when only one limb of the sun is observed, for semidiameter.*

Observed altitudes of either object above the water-line are also to be corrected for the dip by (53) or Tab. XIII. (Bowd.), if the horizon is free; but by (55) or Tab. XVI. (Bowd.), if the horizon is obstructed. The altitude of the celestial object, when not observed simultaneously, may be interpolated from altitudes before and after, by means of the noted times. (Bowd, p. 246.) Or the true altitude may be computed for the local time (Prob. 38 or 39), and the refraction added and the parallax subtracted to obtain the *apparent* altitude.

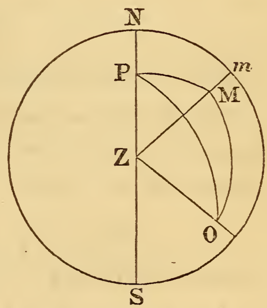

Fig. 52.

* It is best in measuring the distance of the sun from the terrestrial object to use each limb alternately.

Let $h' = 90° - Z\,O$ (Fig. 50), the *apparent* altitude of O,
$H' = 90° - Z\,M$, the *apparent** altitude of M.
$D = M\,O$, the corrected distance.

We have then in the triangle M Z O, the three sides from which $\zeta = M\,Z\,O$, may be found by one of the following formulas :—

1. By Sph. Trig. (164) we have

$$\sin \tfrac{1}{2}\zeta = \sqrt{\frac{\sin \frac{1}{2}(D + H' - h') \sin \frac{1}{2}(D - H' + h')}{\cos H' \cos h'}}$$

or, letting

$$\left.\begin{aligned} d &= H' - h', \\ \sin \tfrac{1}{2}\zeta &= \sqrt{\frac{\sin \frac{1}{2}(D + d) \sin \frac{1}{2}(D - d)}{\cos H' \cos h'}} \end{aligned}\right\} \qquad (230)$$

2. By Sph. Trig. (165).

$$\cos \tfrac{1}{2}\zeta = \sqrt{\frac{\cos \frac{1}{2}(H' + h' + D) \cos \frac{1}{2}(H' + h' - D)}{\cos H' \cos h'}}$$

or, putting

$$\left.\begin{aligned} s &= \tfrac{1}{2}(H' + h' + D) \\ \cos \tfrac{1}{2}\zeta &= \sqrt{\frac{\cos s \cos (s - D)}{\cos H' \cos h'}} \end{aligned}\right\} \qquad (231)$$

(230) is preferable when $\zeta < 90°$; (231), when $\zeta > 90°$.

279. If O is in the true horizon, or its measured altitude above the water line equals the dip, $h = 0$, and the right triangle M m O gives

$$\cos \zeta = \cos M\,O = \cos D \sec H'; \qquad (232)$$

or more accurately when ζ is small (Sph. Trig., 105),

$$\tan \tfrac{1}{2}\zeta = \sqrt{(\tan \tfrac{1}{2}(D + H') \tan \tfrac{1}{2}(D - H'))}. \qquad (233)$$

If the terrestrial object is in the water-line, h' is negative, and equals the dip.

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* The *true* altitude of M is used in finding z, its azimuth.

280. If both objects are in the horizon, or H and h are equal and very small, we have simply

$$\zeta = D. \tag{234}$$

In general the result is more reliable the smaller the inclination of M O to the horizon. If M O is perpendicular to the horizon, the problem is indeterminate by this method.

281. If the terrestrial object presents a vertical line to which the sun's disk is made tangent, the reduction of the observed distance for semidiameter is

$$s' = s \sin \mathrm{M\ O\ Z} \tag{235}$$

and not s, the semidiameter itself. This follows from the sun's diameter through the point of contact, O, being perpendicular to the vertical circle Z O and not in the direction of the distance O M.

As the altitude of the terrestrial object is always very small, we may find M O Z by the formula

$$\cos \mathrm{M\ O\ Z} = \frac{\sin h'}{\sin D'},$$

D' being the unreduced distance.

282. When precision is requisite, the axis of the sextant with which the angular distance is measured must be placed at the station Z; and if the object seen *direct* is sufficiently near, the parallactic correction must be added to the sextant reading. If

$\varDelta$ represent the distance of the object,

d, the distance of the axis from the line of sight or axis of the telescope, this correction is

$$p = \frac{d}{\varDelta} \operatorname{cosec} 1'' = 206265'' \frac{d}{\varDelta}. \tag{236}$$

It is $1'$, when $\varDelta = 3437.75\, d.$

283. If the distance of the terrestrial object and the difference of level above or below the level of the instrument

are known, we may find its angle of elevation, nearly, by the formula

$$\tan h' = \frac{E}{\Delta},$$

Δ being the distance of the object, and

E, its elevation above the horizontal plane of the instrument.

If the object is below that plane, E and h' will have the negative sign.

NOTE.—The horizontal angle between two terrestrial objects may also be found by measuring their angular distance with a sextant, and employing the same formulas (230 to 234) as for a celestial and terrestrial object; H' and h' representing their apparent angles of elevation. Each of these may be found by direct measurement, or from the known distance and the elevation, or depression, from the horizontal plane of the observer. If the two objects are on the same level as the observer, we have simply as in (234)

$$\zeta = D$$

EXAMPLE.

1865, May 16, 5¾ A. M. in lat 38° 15′ N., long. 76° 16′ W.; the angular distance of the sun's centre from the top of a light-house measured by a sextant (☉ to the right of L. H.), 75° 16′ 25″, index cor. −1′ 15″; altitude of $\underline{\odot}$ above the sea-horizon observed at the same time, 10° 18′ 20″, index cor. +2′ 10″; observed altitude of the top of light-house above the water-line, distant 7300 feet, 1° 15′ 20″, index cor., +2′ 10″; height of eye, 20 feet; required the true bearing of the light-house.

From the data we find

☉'s ap. alt.	$H' = 10°\ 31'\ 57''$;	ap. alt. of L. H.	$h' =$	1° 7′ 34″
☉'s true "	$H\ = 10\ \ 27\ \ 7$;	ang. dist.	$D =$	75 15 10
☉'s dec	$+19\ \ 9\ \ 30.$			

Computation (100) *and* (230).

$H = 10° \; 27' \; 7''$	l. sec 0.00726	$H' = 10° \; 31' \; 57''$	l. sec 0.00738
$L = 38 \; 15$	l. sec 0.10496	$h' = 1 \; 7 \; 34$	l. sec 0.00008
$p = 70 \; 50 \; 30$		$d = 9 \; 24 \; 23$	
$2s = 119 \; 32 \; 37$		$D = 75 \; 15 \; 10$	
$s = 59 \; 46 \; 18$	l. cos 9.70196	$\frac{1}{2}(D+d) = 42 \; 19 \; 46$	l. sin 9.82827
$p-s = 11 \; 4 \; 12$	l. cos 9.99184	$\frac{1}{2}(D-d) = 32 \; 55 \; 24$	l. sin 9.73521
	19.80602		19.57094
$\frac{1}{2}Z = 36° \; 53'.0$	l. cos 9.90301	$\frac{1}{2}\zeta = 37° \; 36'.2$	l. sin 9.78547
		$\zeta = 75 \; 12.4$	

⊙'s azimuth $Z =$ N. 73 46.0 E.

True bearing of L. House $(Z-\zeta) =$ N. 1 26.4 W.